ET 644 – Advanced Digital Signal Processing

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session 4 Z-Transform

Outline:

- Z-transform
- Relevant sections in P & M Text:
- Sections 3.1, 3.2, 3.4.3, 3.6.1 - 3.6.4

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- Z-transform

$e^{st}u(t) \overset{CT,LTI,h_a(t)}{\longmapsto} H_a(s)e^{st}$

$H_a(s) = \mathfrak{L}\{h_a(t)\}$

$\mathfrak{L}$ - Laplace Transform

Consider: $x[n] = a^n \overset{LTI,h[n]}{\longmapsto} y[n] = ?$

For all n, $y[n] = \Sigma_{k=-\infty}^{\infty} h[k]x[n-k] = \Sigma_{k=-\infty}^{\infty} h[k]a^{n-k}$

$= \{\Sigma_{k=-\infty}^{\infty} h[k]a^{-k}\}a^n$

- Defining Z-Transform (ZT) of h[n]:

$H(z) = \Sigma_{n=-\infty}^{\infty} h[n]z^{-n}$

$\underbrace{x[n] = a^n}_{-\infty<n<\infty} \overset{LTI,h[n]}{\longmapsto} y[n] = H(a)a^n$

where H(a) is the ZT of h[n] evaluated at z = a.

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Examples:

1. $x[n] = a^n u[n]$
2. $x[n] = -a^n u[-n-1]$
3. $x[n] = a^n u[n] + b^n u[-n-1]$

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Properties of ZT:

• Shifting Property:

$Z\{x[n-k]\} = z^{-k}X(z)$

• Convolution Property:

$Z\{x[n] * h[n]\} = X(z)H(z)$ • See Table 3.2 on p.173 for further properties of ZT.

For an LTI DT system, Y(z) = H(z)X(z)

$$H(z) = Z\{h[n]\} = \frac{Y(z)}{X(z)}$$

• z_0 is a pole of H(z) if $H(z_0) = \infty$.

• z_0 is a pole of H(z) if $H(z_0) = 0$.

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ZT Analysis of LTI SYstem Described by Difference Equations:

- Shifting Property:

$Z\{y[n]\} = Z\{-\Sigma_{k=1}^{N} a_k y[n-k]\} + Z\{\Sigma_{k=0}^{M} b_k x[n-k]\}$
$Y(z) = -\Sigma_{k=1}^{N} a_k z^{-k} Y[z] + \Sigma_{k=0}^{M} b_k z^{-k} X[z]\}$

Convolution Property:

$H(z) = \frac{Y(z)}{X(z)} = \frac{\Sigma_{k=0}^{M} b_k z^{-k}}{1+\Sigma_{k=1}^{N} a_k z^{-k}}$

$= b_0 \frac{z^{-M}}{z^{-N}} \frac{z^M + \frac{b_1}{b_0} z^{M-1} + \cdots + \frac{b_M}{b_0}}{z^N + a_1 z^{N-1} + \cdots + a_N}$

$= b_0 z^{N-M} \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$

- z_k : zeros of H(z)
- p_k: poles of H(z).

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- Assume poles are unique and $M \leq N$ (if not, perform long division first).
  (Keep in mind: $\delta[n-k] \xleftrightarrow{ZT} z^{-k}$)
- Partial Fraction Expansion (PFE):
  $H(z) = A_1 \frac{z}{z-p_1} + A_2 \frac{z}{z-p_2} + \cdots + A_N \frac{z}{z-p_N}$
- where $A_k = \frac{z-p_k}{z} H(z)|_{z=p_k}$

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- Basic inversion result:

$$Z^{-1}\{\frac{z}{z-p_k}\} = \begin{cases} p_k^n u[n], & \text{if ROC} \subset \{|z| > |p_k|\} \\ -p_k^n u[-n-1], & \text{if ROC} \subset \{|z| < |p_k|\} \end{cases}$$

- for repeated poles, see pp.191-193 in P & M Text.

$\Longrightarrow$ Example 3.4.7.

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- if $z_i \neq p_j$, $i = 1, 2, \cdots, M, j = 1, 2, \cdots, N$:
Then $H(z)|_{z=p_j} = \infty$.
- ROC cannot contain a pole • Assume poles ordered as:
$|p_1| \leq |p_2| \leq \cdots \leq |p_N|$
- ROC must lie in an annular region: $|p_k| < |z| < |p_{k+1}|$
- Causality requires h[n] = 0 for $n < 0$
- Implies h[n] must be "right-sided" sequence
- ROC of H(z) must be $|z| > |p_N|$
- P_N: pole with largest magnitude