Session 20

- Forward Linear Prediction (LP)
- Relationship between AR & LP

- What does the recursive relationship mean?

  You can predict future values based on current and past values!
  (or current value based on past values depending on how you view it!)

- We consider a one-step forward Linear Predictor.

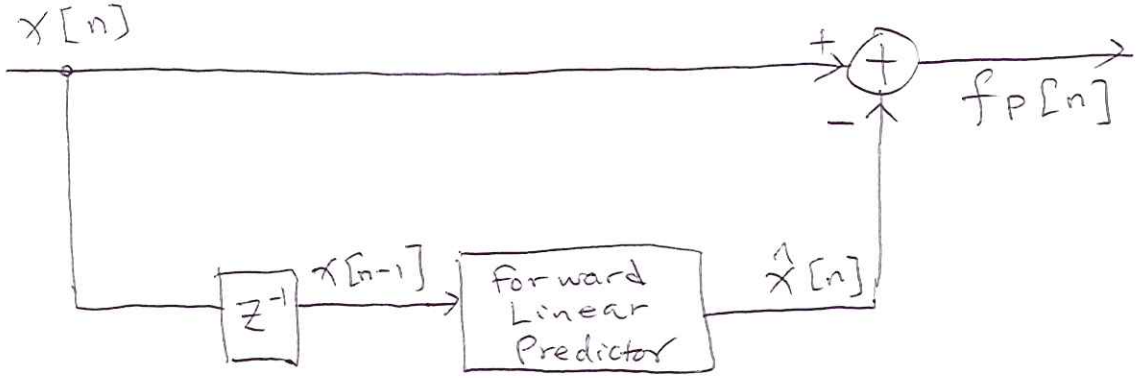
$$\hat{x}[n] = - \sum_{k=1}^{p} a_p(k) x[n-k]$$

  - $\{-a_p(k)\}$ are the weights in the linear Combination.

They are called "prediction coefficients." This predictor is of order p. negative sign — mathematical convenience.

- The difference between the actual value $x[n]$ & the predicted value $\hat{x}[n]$ is the "forward prediction error."

$$f_p[n] = x[n] - \hat{x}[n] = x[n] + \sum_{k=1}^{p} a_p(k)\, x[n-k].$$



Realization of a "prediction error filter."

Filter Realization (direct form) FIR:

$$A_p(z) = \sum_{k=0}^{p} a_p(k)\, z^{-k}$$

where, by definition, $a_p(0)=1$.

- So, we can form the following set of "order-recursive" equations:

$$f_0[n] = g_0[n] = x[n]$$

$$f_1[n] = x[n] + a_p(1)\, x[n-1]$$

$$f_2[n] = x[n] + a_p(1)\, x[n-1] + a_p(2)\, x[n-2]$$

$$\vdots$$

We can also represent the filter as:

$$f_p[n] = \sum_{k=0}^{p} a_p(k)\, x[n-k] \qquad a_p(0) = 1.$$

$\because$ this equation is a convolution, the z-transform gives:

$$F_p(z) = A_p(z)\, X(z).$$

$$A_p(z) = \frac{F_p(z)}{X(z)} = \frac{F_p(z)}{F_0(z)}.$$

$$\mathcal{E} = E\{|e[n]|^2\} = E\{|x[n] - \hat{x}[n]|^2\}$$

$$\hat{x}[n] = -\sum_{k=1}^{p} a_p(k)\, x[n-k]$$

$$\Rightarrow \quad \mathcal{E} = E\left\{\left|x[n] + \sum_{k=1}^{p} a_p(k)\, x[n-k]\right|^2\right\}$$

To find $\mathcal{E}_p^{min}$:

$$\frac{\partial \mathcal{E}}{\partial a_p(\ell)} = E\left\{\frac{\partial}{\partial a_p(\ell)}\left[x[n] + \sum_{k=1}^{p} a_p(k)\, x[n-k]\right]^2\right\}$$

$$= E\left\{2 \cdot \left[\underbrace{x[n] + \sum_{k=1}^{p} a_p(k) \cdot x[n-k]}_{e[n]}\right] \cdot x[n-\ell]\right\} = 0$$

$$\Rightarrow \quad E\left\{2 \cdot x[n]\, x[n-\ell] + 2 \sum_{k=1}^{p} a_p(k)\, x[n-k] \cdot x[n-\ell]\right\} = 0$$

$$\Rightarrow \quad r_{xx}[\ell] = -\sum_{k=1}^{p} a_p(k) \cdot r_{xx}[\ell-k]$$

$$\ell = 1, 2, \cdots, p.$$

$x[n]$ is an AR(p) process, the above equations are the same as that relating $r_{xx}(\ell)$, $\ell = 0, 1, \cdots, p$ to a_k, $k = 1, 2, \cdots, p$. $\Rightarrow a_p(k) = a_k$, $k = 1, 2, \cdots, p$.

- minimum prediction error:

$$\mathcal{E}_m^{min} = E\left\{ \left[x[n] + \sum_{k=1}^{m} a_m x[n-k]\right] \left[x^*[n] + \sum_{k=1}^{m} a_m^*(k) \cdot x^*[n-k]\right] \right\}$$

<u>End result</u>: $$\mathcal{E}_m^{min} = r_{xx}[0] + \sum_{k=1}^{m} a_m(k) \cdot \underset{\uparrow \atop r_{xx}[-k]}{r_{xx}^*[k]}$$

- This set of equations may be solved efficiently via Levinson-Durbin algorithm (Sec. 11.3.1)
- L-D algorithm sequentially solves Yule-Walker eqns. for progressively higher predictor orders.

$r_{xx}[0]\, a_1(1) = -r_{xx}[1]$ ← 1^{st} order

$$\begin{bmatrix} r_{xx}[0] & r_{xx}^*[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a_2(1) \\ a_2(2) \end{bmatrix} = - \begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \end{bmatrix}$$

$\vdots$

Outline of L-D algorithm:

- Initialization: $a_0(0) = 0$ ← 0^{th} order predictor

$$\mathcal{E}_0^{min} = E\{(x[n]-0)^2\} = r_{xx}[0]$$

- for $m = 1, 2, \ldots, p$

$$a_m(m) = -\frac{r_{xx}[m] + \sum_{k=1}^{m-1} a_{m-1}(k)\, r_{xx}[m-k]}{\mathcal{E}_{m-1}^{min}}$$

- for $k = 1, \ldots, m-1$:

$$a_m(k) = a_{m-1}(k) + a_m(m)\, a_{m-1}^*(m-k)$$

- end
- $\mathcal{E}_m^{min} = \mathcal{E}_{m-1}^{min}\{1 - |a_m(m)|^2\}$
- end

- The mean-square value of the error is:

$$\mathcal{E}_p^f = E\left[|f_p[n]|^2\right]$$

- And MMSE: (Minimum Mean Square Error):

$$\min\left[\mathcal{E}_p^f\right] = r_{xx}[0] + \sum_{k=1}^{p} a_p(k)\, r_{xx}(-k).$$

See supplemental notes

<u>Numerical Example:</u>

Given: $r_{xx}[0]=1$, $r_{xx}[1]=\frac{1}{2}$, $r_{xx}[2]=\frac{1}{8}$.

for an AR(2) process.

1. Determine AR model parameters a_1 & a_2 and σ_w^2 of iid noise input to the 2-pole filter that generated $x[n]$, an AR(2) process.
2. Determine $r_{xx}[3]$
3. Closed form expression for $S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m]\, e^{-j\omega m}$

Solution:
$$a_1(1) = -\frac{r_{xx}[1]}{\mathcal{E}_0^{min}} = -\frac{r_{xx}[1]}{r_{xx}[0]} = \frac{-\frac{1}{2}}{1} = \frac{1}{2}$$

- minimum prediction error:

$$\mathcal{E}_1^{min} = \mathcal{E}_0^{min}\{1-|a_1(1)|^2\} = r_{xx}[0]\{1-(-\tfrac{1}{2})^2\} = 3/4$$

- 2^{nd} order predictor:

$$a_2(2) = -\frac{\{r_{xx}[2] + a_1(1)\, r_{xx}[1]\}}{\mathcal{E}_1^{min}}$$

$$a_2(2) = -\frac{\{\frac{1}{8} - \frac{1}{2}(\frac{1}{2})\}}{3/4} = \frac{1}{6} = a_2$$

AR model parameters

$$a_2(1) = a_1(1) + a_2(2)\, a_1^*(1)$$
$$= a_1(1)\{1 + a_2(2)\} = (-\tfrac{1}{2})\{1+\tfrac{1}{6}\} = -\frac{7}{12} = a_1$$

- $$\mathcal{E}_2^{min} = \mathcal{E}_1^{min}\{1-|a_2(2)|^2\}$$
$$= \frac{3}{4}\{1-(\tfrac{1}{6})^2\} = \frac{35}{48} = \sigma_w^2$$

· Check:

$$\sigma_w^2 = r_{xx}[0] + \sum_{k=1}^{2} a_k r_{xx}^*[k]$$

$$= 1 + \left(\frac{-7}{12}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{8}\right) = \frac{35}{48} \quad \checkmark$$

· check:

$$\begin{bmatrix} r_{xx}[0] & r_{xx}^*[1] \\ r_{xx}[1] & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a_2(1) \\ a_2(2) \end{bmatrix} = -\begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{7}{12} \\ \frac{1}{6} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{8} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

· 2.

$$r_{xx}[3] = -\sum_{k=1}^{p} a_k r_{xx}[m-k] = -a_1 r_{xx}[2] - a_2 r_{xx}[1]$$

$$= \frac{7}{12}\left(\frac{1}{8}\right) - \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = -\frac{1}{96}$$

3.

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m] e^{-j\omega m} = \frac{\sigma_w^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-jk\omega}\right|^2}$$

$$= \frac{35/48}{\left|1 - \frac{7}{12} e^{-j\omega} + \frac{1}{6} e^{-j2\omega}\right|^2}.$$