Session 19

Outline

- Extrapolation of the autocorrelation for a SoS signal

- Basics of AutoRegressive (AR) spectral estimation (most widely used parametric spectral estimation technique)

- Problem with non-parametric spectral estimators: inherently assume $r_{xx}[m] = 0$ for $m > M \leftarrow$ Block length where $M \leq N-1$

- In fact, for SoS (w/o noise)

$$r_{xx}[m] = \sum_{i=1}^{P} P_i e^{j\omega_i m}$$

does NOT decay to zero as $m \to \infty$.

- in general, sharp spectral peaks in $S_{xx}(\omega)$ (true spectrum) imply $r_{xx}[m]$ takes a "long time" to decay to zero as $m \to \infty$.

- interesting observation re: $r_{xx}[m]$ for SoS model:

$\Rightarrow$ it satisfies recursive relationship:

$$r_{xx}[m] = -\sum_{k=1}^{P} a_k \, r_{xx}[m-k] \quad \text{for } m \geq P$$

- this means, if there are P sinewaves, the true autocorrelation value of lag value m is a linear combination of $r_{xx}[m-1]$, $r_{xx}[m-2]$, ..., all the way to P, $r_{xx}[m-p]$.

- where a_k, $k=1,2,\ldots,P$, are the coefficients of the polynomial $(z-e^{j\omega_1})(z-e^{j\omega_2})\cdots(z-e^{j\omega_P})$

$$= z^P + a_1 z^{P-1} + \cdots + a_{P-1} z + a_P$$

a root of the polynomial

$$\Rightarrow 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{P-1} z^{-(P-1)} + a_P z^{-P} = 0 \quad \text{for } z_i = e^{j\omega_i}$$

$$1+a_1 e^{j\omega_i(-1)} + a_2 e^{j\omega_2(-2)} + \cdots + a_{p-1} e^{j\omega_i(-(p-1))} + a_p e^{j\omega_i(-p)} = 0.$$

- multiply both sides by $P_i e^{j\omega_i m}$ $\quad (i=1,2,\ldots,p)$

- Sum over all i (since $\sum 0 = 0$)

$$\sum_{i=1}^{p} P_i e^{j\omega_i m} + a_1 \sum_{i=1}^{p} P_i e^{j\omega_i(m-1)} + \cdots + a_{p-1} \sum_{i=1}^{p} P_i e^{j\omega_i(m-(p-1))}$$
$$+ a_p \sum_{i=1}^{p} P_i e^{j\omega_i(m-p)} = 0.$$

Without noise, $\displaystyle\sum_{i=1}^{p} P_i e^{j\omega_i m} = r_{xx}[m]$

$$\sum_{i=1}^{p} P_i e^{j\omega_i(m-1)} = r_{xx}[m-1]$$

$$\vdots$$

$$\sum_{i=1}^{p} P_i e^{j\omega_i(m-p)} = r_{xx}[m-p]$$

- ultimately, $\displaystyle r_{xx}[m] = -\sum_{k=1}^{p} a_k r_{xx}[m-k]$ for $m \geq p$.

- <u>QED</u>.

- Thus, if:
  - if we estimate $r_{xx}[m]$ for $m = 0, 1, \ldots, P$ **AND**
  - determine the coefficients a_k, $k = 1, 2, \ldots, P$.
- We could extrapolate $r_{xx}[m]$ for all m, thereby averting a windowing effect
- ostensibly use recursion to determine $r_{xx}[m]$ for $m \geq P$ and form $S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m] e^{-j\omega m}$
- However, in case of SoS, typically interested in frequencies, ω_i, which may be determined from roots of $z^P + a_1 z^{P-1} + \cdots + a_{P-1} z + a_P$.

- Note: $r_{xx}[-m] = r_{xx}^*[m]$

- How do we estimate a_k, $k =$ [redacted] $, 2, \cdots, p$.

- $$r_{xx}[m] = -\sum_{k=1}^{p} a_k\, r_{xx}[m-k]$$

- Write out p equations with p unknowns $\Rightarrow m = 1, 2, \cdots, p$.

$$\begin{bmatrix} r_{xx}[0] & r_{xx}[-1], & \cdots & r_{xx}[-(p-1)] \\ r_{xx}[1] & r_{xx}[0] & \cdots & r_{xx}[-(p-2)] \\ \vdots & \vdots & \vdots & \vdots \\ r_{xx}[p-1] & r_{xx}[p-2] & \cdots & r_{xx}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = - \begin{bmatrix} r_{xx}[1] \\ r_{xx}[2] \\ \vdots \\ r_{xx}[p] \end{bmatrix}$$

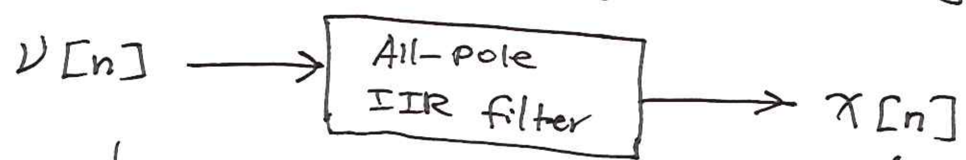
$$\Rightarrow \quad \underset{p\times p}{\underline{R}}\;\underset{p\times 1}{\underline{a}} = -\underset{p\times 1}{\underline{r}}$$

(See SoSextrap.m)

- these observations for the SoS model are the basis for the following spectral estimation techniques:
  - AutoRegressive Method (AR)
  - MUSIC (MUltiple SIgnal Classification) sect. 12.5.3
  - ESPRIT (Estimation of Signal Parameters by Rotational Invariance Techniques) Sect. 12.5.4

- AR Spectral Estimation

- for sole purpose of matching the spectral density of a WSS random process, $x[n]$, we model $x[n]$ as having been generated as:



where $r_{\nu\nu}[m] = \sigma_w^2 \delta[m]$.

$x[n]$: AR(p) Process

- IIR means it's a difference equation

- $$x[n] = -\sum_{k=1}^{p} a_k x[n-k] + \nu[n]$$

- Can show that $r_{xx}[m]$ satisfies:

$$r_{xx}[m] = -\sum_{k=1}^{p} a_k r_{xx}[m-k] + \sigma_w^2 \delta[m]$$

- very similar to what we have in SoS.
- A signal generated by passing a white noise through an all-pole IIR filter. Such a signal satisfies the recursive relation.

- in frequency domain:

$$\mathbb{X}(\omega) = H(\omega)\, \mathbb{V}(\omega)$$

- where $$H(\omega) = \frac{1}{1 + \sum_{k=1}^{p} a_k e^{-jk\omega}}$$

- from basis random process theory

$$S_{xx}(\omega) = |H(\omega)|^2 S_{\nu\nu}(\omega).$$

where $$S_{\nu\nu}(\omega) = \text{DTFT}\{r_{\nu\nu}[m]\}$$
$$= \text{DTFT}\{\sigma_w^2 \delta[m]\} = \sigma_w^2 \quad \forall \omega$$

- thus, $$S_{xx}(\omega) = \text{DTFT}\{r_{xx}[m]\}$$

$$= \frac{\sigma_w^2}{\left| 1 + \sum_{k=1}^{p} a_k e^{-jk\omega} \right|^2}.$$