Session 18

Outline

- Non-parametric estimation of autocorrelation and power spectrum (Sect. 12.1, 12.2.3)
- Sum of Sinewaves model (Sect. 12.5.2)
- DT random process (Appendix A)

Couse Eval

5 pts.

- Parametric Spectral Estimation
  - with emphasis on harmonic retrieval methods
    - i.e. estimating frequencies and amplitudes of superimposed sinewaves (or geometric sequences) embedded in noise.
  - Parametric = model-based
  - Why parametric?
    - to achieve higher-resolution of sharp spectral peaks than that achieved with non-parametric spectral estimation
    - the latter is primarily Wiener-Kinchen Theorem

Wiener - Kinchen Theorem

- Power or energy density spectrum of a DT signal $x[n]$ is:

$$S_{xx}(\omega) = |X(\omega)|^2 \qquad \text{where:}$$

$$x[n] \xleftrightarrow{DTFT} X(\omega)$$

- $$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} r_{xx}[m] e^{-jm\omega}$$

Where: $$r_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]\, x^*[n-m]$$

- Stochastic Case:

$$S_{xx}(\omega) = E\{|X(\omega)|^2\}$$

- $$r_{xx}[m] \xleftrightarrow{DTFT} S_{xx}(\omega)$$
- where $$r_{xx}[m] = E\{x[n]\, x^*[n-m]\}$$

  $E\{\cdot\}$ — Expectation operator (See Appendix A)
- $x[n]$ — Assumed to be wide-sense stationary (WSS)

- In practical cases, given N samples of a WSS random process (RP) $x[n]$, $n = 0, 1, \cdots, N-1$.

- Autocorrelation is estimated as:

$$\hat{r}_{xx}[m] = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x[n]\, x^*[n-m] \qquad \left(\text{Sort of like taking } E\{\cdot\}\right)$$

for $m = 0, 1, \cdots, N-1$.

~~Simple example~~

$\cancel{x[n] = \{0, 1, 2\} \Rightarrow N = 3 \quad \hat{r}_{xx}[m] = \frac{1}{3-m} \sum_{n=0}^{3-m-1} x[n]\, x^*[n-m]}$

$\cancel{\hat{r}_{xx}[0] = \frac{1}{3} \sum_{n=0}^{2} x[n]\, x^*[n] = \frac{1}{3}(0+1+4) = \frac{5}{3}}$

$\cancel{\hat{r}_{xx}[1] = \frac{1}{2} \sum_{n=0}^{1} x[n]\, x^*[n-1] = \frac{1}{2}(0+0)}$

- Note: upper-limit in sum
  - for a given m, there are $(N-m)$ pairs of samples separated by m units of DT to average over in order to:
    - achieve uncorrelatedness/independence amongst different constituent signals comprising $x[n]$
    - average out the effects of noise

$\hat{r}_{xx}[1] \cdots$

$\hat{r}_{xx}[2] \cdots$

Also — pictorial presentation

Simple example of estimated autocorrelation:

$$x[n] = \{\underset{\substack{\uparrow \\ n=0}}{0}, 1, 2, 3\}. \implies N = 4$$

$$\hat{r}_{xx}[m] = \frac{1}{4-m} \sum_{n=0}^{4-m-1} x[n]\, x^*[n-m]$$

In this case, $x[n] = x^*[n]$

$m=0$, $$\hat{r}_{xx}[0] = \frac{1}{4} \sum_{n=0}^{3} x[n]\, x^*[n] = \frac{1}{4}(0\times0 + 1\times1 + 2\times2 + 3\times3)$$
$$= \frac{1}{4} \times 14 = \frac{7}{2}$$

$x[n]: \{0, 1, 2, 3\}$

$x^*[n]: \{0, 1, 2, 3\}$

$m=1$, $$\hat{r}_{xx}[1] = \frac{1}{3} \sum_{n=0}^{2} x[n]\, x[n-1]$$
$$= \frac{1}{3}(0\times0 + 1\times0 + 2\times1 + 3\times2) = \frac{8}{3}$$

$x[n]: \{0, 1, 2, 3\}$

$x[n-1]: \{\underset{\substack{\uparrow \\ n=0}}{0}, 1, 2, 3\}$

$m=2$, $$\hat{r}_{xx}[2] = \frac{1}{2} \sum_{n=0}^{1} x[n]\, x[n-2]$$
$$= \frac{1}{2}(0\times 0 + 2\times0 + 3\times1) = \frac{3}{2}$$

$x[n]: \{0, 1, 2, 3\}$

$x[n-2]: \{\underset{\substack{\uparrow \\ n=0}}{0}, 1, 2, 3\}$

$m=3$, $$\hat{r}_{xx}[3] = \frac{1}{1} \sum_{n=0}^{0} x[n]\, x^*[n-3]$$
$$= 0.$$

$x[n]: \{0, 1, 2, 3\}$

$x[n-3]: \{\underset{\substack{\uparrow \\ n=0}}{0}, 1, 2, 3\}$

Case Study: Sum of Sinewaves (SoS) in noise

$$x[n] = \sum_{i=1}^{p} A_i e^{j(\omega_i n + \phi_i)} + \nu[n]$$

- A_i, ω_i — deterministic but unknown
- ϕ_i — independently identically distributed (i.i.d.) uniform random variables over $[0, 2\pi)$.
- $\nu[n]$ — Contribution from noise

$$f_\Phi(\varphi) = \begin{cases} \frac{1}{2\pi} & \varphi \in [0, 2\pi) \\ 0 & \text{else} \end{cases}$$

General case:
$X \sim U[a, b]$.
$f_X(x) = \frac{1}{b-a} \mathbb{1}_{[a,b]}$.

- Assumptions:

i.) zero-mean: $E\{\nu[n]\} = 0$

ii) uncorrelated with signal / independent

$$E\{\nu[n] s^*[n-m]\} = 0 \qquad \forall m$$

where $$s[n] = \sum_{i=1}^{p} A_i e^{j\phi_i} e^{j\omega_i n}$$

iii) $\nu[n]$ is a DT WSS rp.

$$\gamma_{\nu\nu}[m] = E\{\nu[n]\,\nu^*[n-m]\}$$

- Typically assume white noise:

$$\gamma_{\nu\nu}[m] = \sigma_w^2 \cdot \delta[m]$$

- $E\{\cdot\}$ is a linear operator:

$$E\left\{\sum_{i=1}^{Q} a_i x_i\right\} = \sum_{i=1}^{Q} a_i E\{x_i\}.$$

- "True" autocorrelation for SoS in noise (vs. estimated):

$$\gamma_{xx}[m] = E\{x[n]\,x^*[n-m]\}$$

$$= E\left\{\sum_{i=1}^{P} A_i e^{j\phi_i} e^{j\omega_i n} \sum_{\ell=1}^{P} A_\ell e^{-j\phi_\ell} e^{-j\omega_\ell (n-m)}\right\} + \cdots$$

Three terms resolted from multiplications

$$= \sum_{i=1}^{P}\sum_{\ell=1}^{P} A_i A_\ell e^{j\omega_i n} e^{-j\omega_\ell (n-m)} \cdot \underbrace{E\{e^{j\phi_i} e^{-j\phi_\ell}\}}_{\delta[i-\ell]}.$$

$$+\underbrace{E\{\nu[n]\,s^*[n-m]\}}_{=0} + \underbrace{E\{s[n]\,\nu^*[n-m]\}}_{=0} + \underbrace{E\{\nu[n]\,\nu^*[n-m]\}}_{=\sigma_w^2\,\delta[m]}.$$

- due to i.i.d assumption, $\forall\ i \neq l$: $\left(\begin{matrix}\text{if } i = l, \\ e^{j\phi_i} \cdot e^{-j\phi_l} = 1\end{matrix}\right)$.

$$E\{e^{j\phi_i} \cdot e^{-j\phi_l}\} = E\{e^{j\phi_i}\}\, E\{e^{-j\phi_l}\}$$

$$E\{e^{j\phi_i}\} = \int_0^{2\pi} \frac{1}{2\pi} e^{j\phi}\, d\phi \qquad \left(\phi_i \sim U[0, 2\pi],\ f_\phi(\varphi) = \frac{1}{2\pi}\right)$$

$$= -j\frac{1}{2\pi} e^{j\phi}\Big|_0^{2\pi} = \frac{1}{2\pi j}\{1 - 1\} = 0.$$

Same for $E\{e^{-j\phi_l}\}$.

- Thus: $$r_{xx}[m] = \sum_{i=1}^{P} P_i e^{j\omega_i m} + \sigma_w^2 \delta[m]$$

Where: $P_i = A_i^2$, $i = 1, 2, \cdots, P$. $\forall m$

(only when $i = l$, $r_{xx}[m] \neq 0$, else, $r_{xx}[m] = 0$).

- Compare to time-averaged estimate of autocorrelation
- Substitute SoS in noise model into:

$$\hat{r}_{xx}[m] = \left\{\sum_{n=0}^{N-m-1} x[n]\, x^*[n-m]\right\} \frac{1}{N-m}$$

- $$r_{xx}[m] = \sum_{i=1}^{P} \sum_{\ell=1}^{P} A_i A_\ell \, e^{j(\phi_i - \phi_\ell)} e^{j\omega_\ell m} \cdot \frac{1}{N-m} \sum_{n=0}^{N-m-1} e^{j(\omega_i - \omega_\ell)n}$$

$$+ \frac{1}{N-m} \sum_{n=0}^{N-m-1} s[n] \, \nu^*[n-m]$$

$$+ \frac{1}{N-m} \sum_{n=0}^{N-m-1} \nu[n] \, s^*[n-m]$$

$$+ \frac{1}{N-m} \sum_{n=0}^{N-m-1} \nu[n] \, \nu^*[n-m]$$

- for lag value m, need to average over enough DT samples pairs separated by m so that:

$$\frac{1}{N-m} \sum_{n=0}^{N-m-1} e^{j(\omega_i - \omega_\ell)n}$$

$$= \frac{1}{N-m} \, \frac{1 - e^{j(\omega_i - \omega_\ell)(N-m)}}{1 - e^{j(\omega_i - \omega_\ell)}}$$

(Cont'd on next page)

$$= \frac{1}{N-m} \underbrace{\frac{\sin\left[\frac{N-m}{2}(\omega_i - \omega_\ell)\right]}{\sin\left[\frac{1}{2}(\omega_i - \omega_\ell)\right]}}_{\lim_{N\to\infty} \; = \delta[i-\ell]}$$

for fixed $m<\infty$.

$$= \frac{1}{\sin\left[\frac{1}{2}(\omega_i-\omega_\ell)\right]} \cdot \frac{\sin\left[\frac{N-m}{2}(\omega_i-\omega_\ell)\right]}{N-m} \qquad (*)$$

- $\frac{\sin\{\cdot\}}{\sin\{\cdot\}}$ term acts like a "sinc" with fixed m when $N \to \infty$

- It is demonstrated that when $N \to \infty$

$$\hat{r}_{xx}[m] = r_{xx}[m] \quad \text{(estimate = true)}$$

- $(*)$ expression makes it more look like a "Sinc" function. when $\omega_i - \omega_\ell$ small, $\sin\frac{(\omega_i-\omega_\ell)}{2} \approx \frac{\omega_i-\omega_\ell}{2}$.

- as lag value m increases, less number of pairs to average over

- estimates of $r_{xx}[m]$ become increasingly unreliable as "m" increases.

- General form of non-parametric spectral Estimator: $$\hat{S}_{xx}(\omega) = \sum_{m=-(N-1)}^{N-1} \hat{r}_{xx}[m]\, w[m]\, e^{-j\omega m}$$

  - where $w[m]$ is a real-valued, symmetric ($w[-m] = w[m]$), tapered ($w[m+1] \leq w[m]$), and $W(\omega) = \text{DTFT}\{w[m]\} \geq 0 \quad \forall\, \omega$.

- e.g. Bartlett (Triangle) Window:

- $w[m] = \dfrac{N - |m|}{N}$, $m = 0, 1, \cdots, M$ where $M \leq N-1$

  $= 0$ otherwise

- Expected value of non-parametric spectral estimator (See Sect. 12.2.3)

- $$E\{\hat{S}_{xx}(f)\} = \sum_{m=-(N-1)}^{N-1} E\{\hat{r}_{xx}[m]\} w[m] e^{-j\omega m}$$

$$= \sum_{m=-\infty}^{\infty} r_{xx}[m]\, w[m]\, e^{-j\omega m}$$

$$= S_{xx}(\omega) \circledast W(\omega)$$

true spectrum   periodic convolution

See. p.303 of P&M Text

- Convolution operation with $W(\omega)$ causes "smearing" of fine spectral details
<u>and</u> "ringing" (due to sidelobes) — noisy background that can mask weak signal components.

- Return to SoS case study:

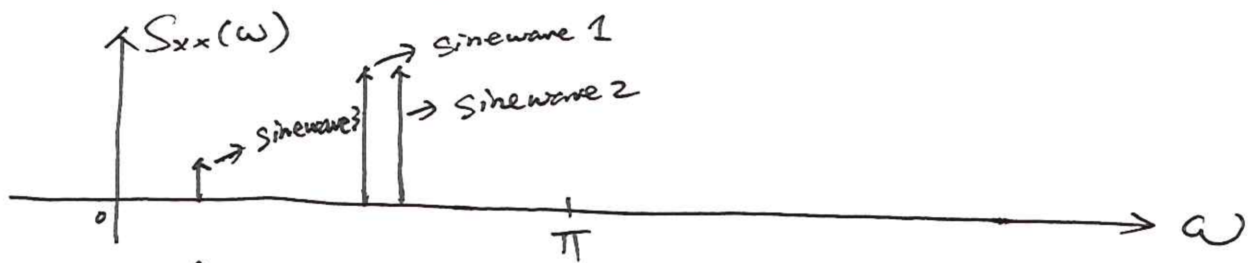
$$S_{xx}(f) = \text{DTFT}\{r_{xx}[m]\}$$

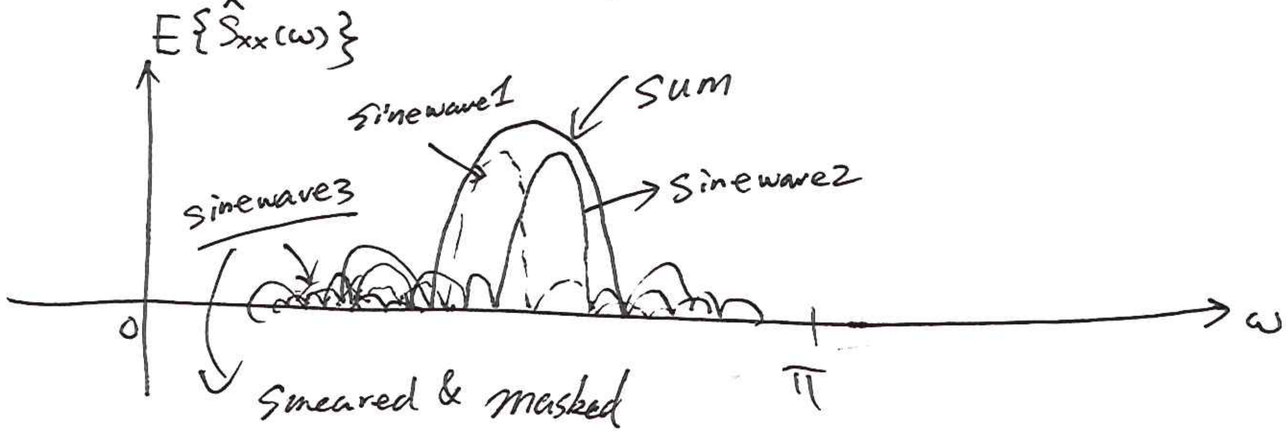
$$= \sum_{i=1}^{P} P_i \; 2\pi \delta_a(\omega - \omega_i) + \sigma_w^2 . \qquad \text{for } |\omega| < \pi$$

Dirac-delta fcn.

$$E\{\hat{S}_{xx}(\omega)\} = \sum_{i=1}^{P} P_i \, W(\omega - \omega_i) + \sigma_w^2 \, w[0]$$

- From previous page, we can see:
  - Estimated spectrum ($E\{\cdot\}$) is a good estimate of the true spectrum
  - Because of ~~wind~~ window effects (sidelobes) causing "smearing" & "masking" of small signal components.
  - Example: $P=3$:





by sidelobes of sinewave 1 & 2.