Session 17

Outline

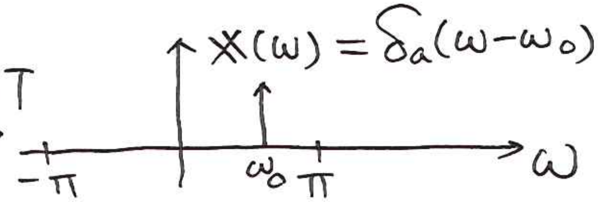
- Analysis of windowing
  (Sect. 8.2.2, Sect. 12.1.1)

- Estimation of autocorrelation and power spectrum for random signals – Sect. 12.1.2

- Analysis of windowing effects

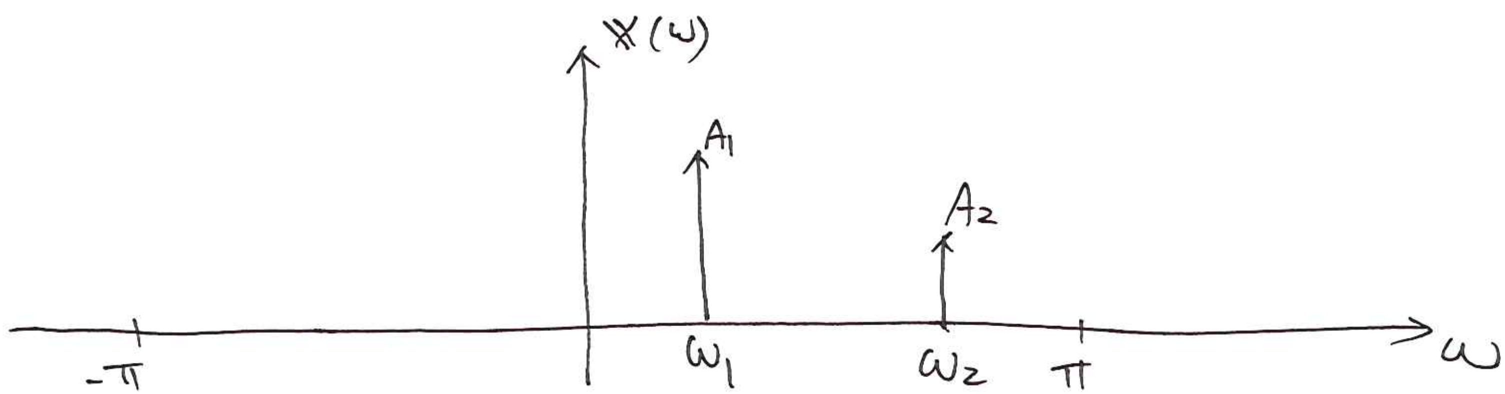
- Only have finite length blocks of data in practice
- If data block is extracted from longer stream of data $\Rightarrow$ have effectively multiplied by rectangular window (lots of high frequency terms, lots of sidelobes, serious ringing).
- To diminish effects of truncation, typically employ tapered window in practice

- analysis of effects of windowing with illustrative example of a Sum of Sinewaves (SoS)

- Recall: $x[n] = e^{j\omega_0 n} \quad \xleftrightarrow{DTFT}$  $\quad -\infty < n < \infty$

thus, if $x[n] = A_1 e^{j\omega_1 n} + A_2 e^{j\omega_2 n} \quad -\infty < n < \infty$

$\xleftrightarrow{DTFT} \; X(\omega) = A_1 \delta(\omega-\omega_1) + A_2 \delta(\omega-\omega_2)$



• Truncating: $\bar{x}[n] = x[n]\, w[n]$

• where: $w[n] = u[n] - u[n-M]$

(Symmetric FIR centered at $\frac{2n-M}{2}$)

Then: $$W(\omega) = \underbrace{e^{-j\left(\frac{M-1}{2}\right)\omega}}_{\text{Linear Phase term.}} \frac{\sin\left(\frac{M}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}$$

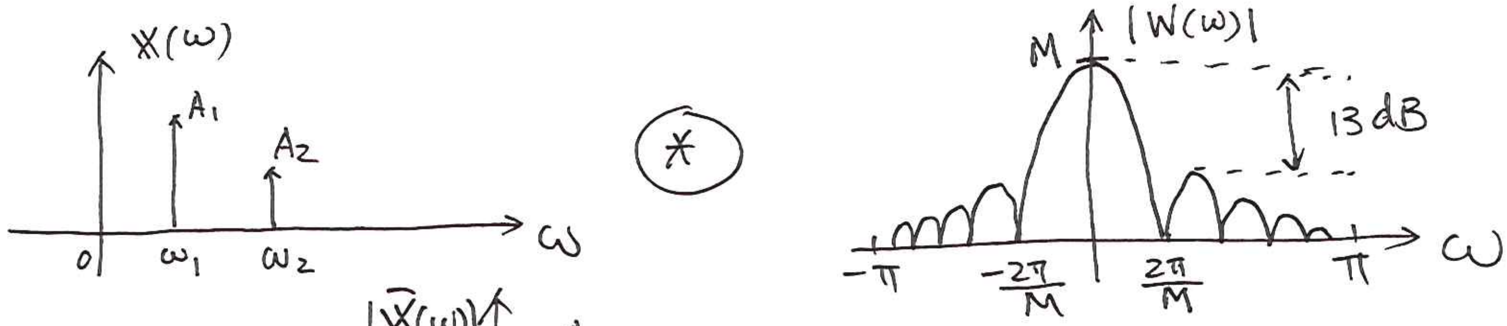
— Derivation in class ①

$$\bar{X}(\omega) = X(\omega) \circledast W(\omega)$$

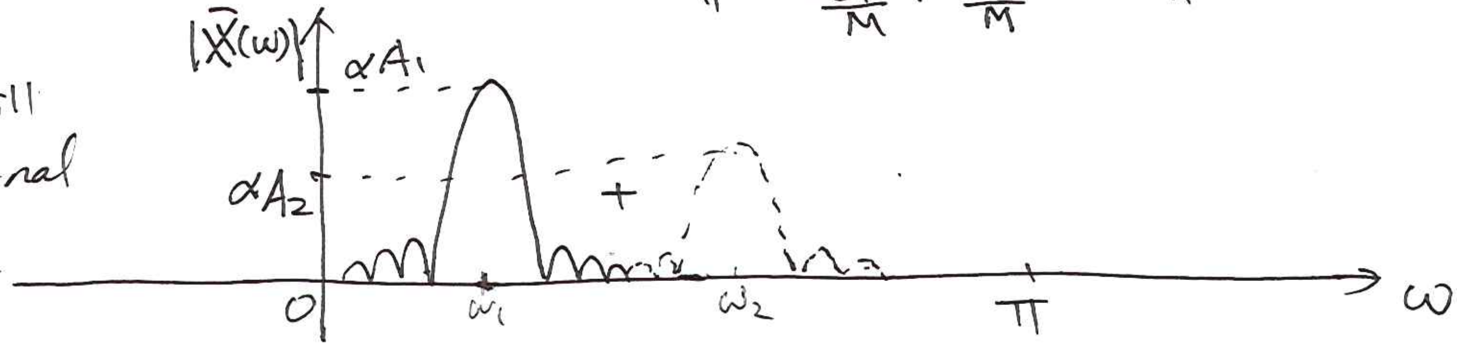
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(u)\, W(\omega-u)\, du$$

periodic convolution

(Recall: DTFT is periodic with period $= 2\pi$).



Amplitudes will be proportional to A_1 & A_2



Refer to ~~[illegible]~~ SINEWINEG.M

- Mainlobe width of $W(\omega)$ affects resolution of sinewaves closely-spaced in frequency
- Sidelobes of $W(\omega)$ caused by stronger sinusoidal component can "mask" presence of weaker sinusoidal components

- Examples of Tapered windows

- Sine window (one-half of a sinewave)

  → DSP book: $w[n] = \sin\left(\frac{\pi}{M-1} n\right)$ $\quad n = 0, 1, \cdots, M-1.$

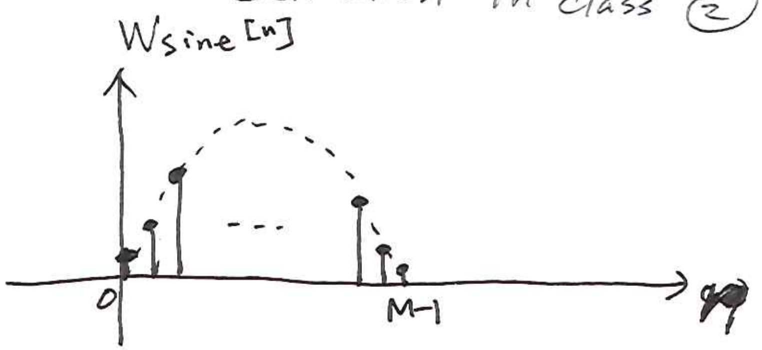
  - alternative format (not equivalent):

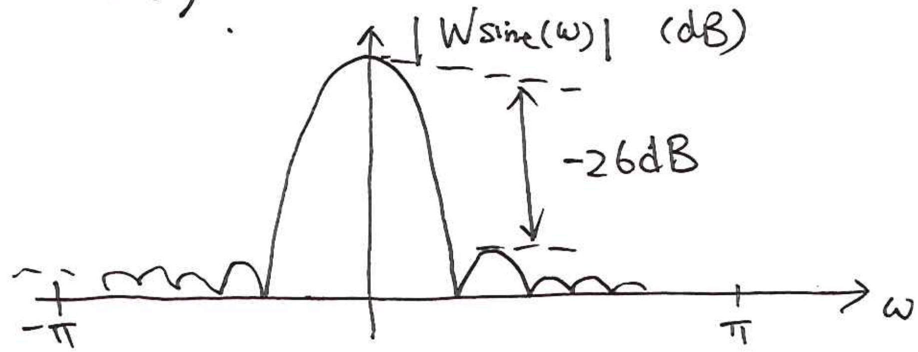
$$W[n] = \sin\left(\frac{\pi}{M}(n+0.5)\right).$$

$$n = 0, 1, \cdots, M-1.$$

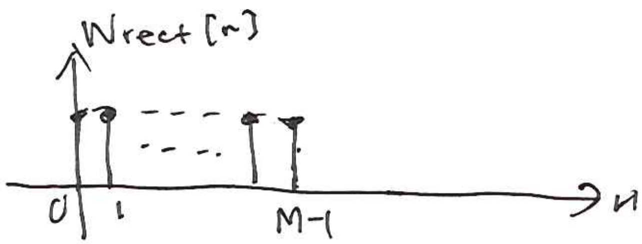
$$= \cos\left(\frac{\pi}{M}\left(n - \frac{M-1}{2}\right)\right)$$

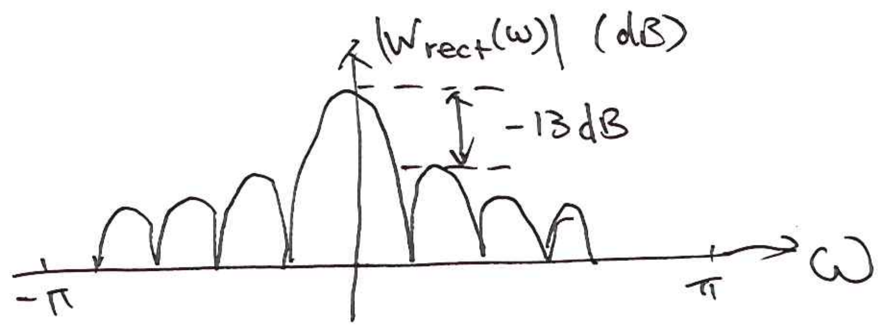
— Derivation in class 2





- Compared to



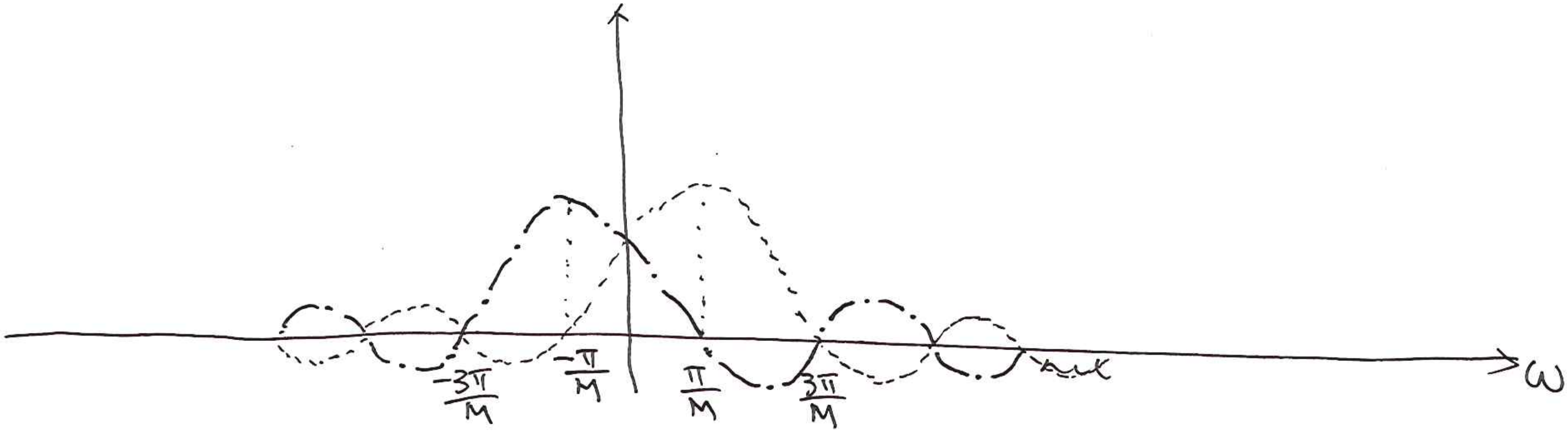


- Analyze sidelobe reduction effect.
- recall: $$e^{j\omega_0 n} w[n] \xleftrightarrow{DTFT} W(\omega-\omega_0)$$
- $$w_{sine}[n] = \sin\left(\frac{\pi}{M}\left(n+\frac{1}{2}\right)\right) w_{rect}[n]$$
$$= \frac{1}{2j} e^{j\frac{\pi}{2M}} e^{j\frac{\pi}{M}n} w_{rect}[n]$$
$$- \frac{1}{2j} e^{-j\frac{\pi}{2M}} e^{-j\frac{\pi}{M}n} \cdot w_{rect}[n]$$

- $$W_{sine}(\omega) = \frac{1}{2} e^{j\left(\frac{\pi}{2M}-\frac{\pi}{2}\right)} \frac{\sin\left(\frac{M}{2}\left(\omega-\frac{\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega-\frac{\pi}{M}\right)\right)} e^{-j\left(\frac{M-1}{2}\right)\left(\omega-\frac{\pi}{M}\right)}$$
$$+ \frac{1}{2} e^{j\left(-\frac{\pi}{2M}+\frac{\pi}{2}\right)} \frac{\sin\left(\frac{M}{2}\left(\omega+\frac{\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega+\frac{\pi}{M}\right)\right)} e^{-j\left(\frac{M-1}{2}\right)\left(\omega+\frac{\pi}{M}\right)}$$

$$= \frac{1}{2} e^{-j\left(\frac{M-1}{2}\right)\omega} \left\{ \frac{\sin\left(\frac{M}{2}\left(\omega-\frac{\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega-\frac{\pi}{M}\right)\right)} + \frac{\sin\left(\frac{M}{2}\left(\omega+\frac{\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega+\frac{\pi}{M}\right)\right)} \right\}$$

— Derivation in class 3



- Sidelobes are 180° out of phase
- When summed, they cancel – hence sidelobes reduction

$$\cancel{③\ W_{sine}(\omega) = DTFT\left\{ \sin\left(\frac{\pi}{M}\cdot\left(n+\frac{1}{2}\right)\right) W_{rect}[n] \right\}}$$

$$\cancel{= DTFT\left\{ \cos\left(\frac{\pi}{M}\left(n-\frac{M-1}{2}\right)\right) W_{rect}[n] \right\}}$$

③ $W_{sine}[n] = \sin\left(\frac{\pi}{M}\left(n+\frac{1}{2}\right)\right)\cdot W_{rect}[n]$

$$= \frac{1}{2j} e^{j\frac{\pi}{2M}} \cdot e^{j\frac{\pi}{M}n} \cdot W_{rect}[n] - \frac{1}{2j} e^{-j\frac{\pi}{2M}} \cdot e^{-j\frac{\pi}{M}n} \cdot W_{rect}[n]$$

$\because \quad DTFT\{W_{rect}[n]\} = \dfrac{\sin\frac{M}{2}\omega}{\sin\frac{1}{2}\omega} \cdot e^{-j\omega\left(\frac{M-1}{2}\right)}$

$\& \quad e^{j\omega_0 n}\cdot W[n] \xleftrightarrow{DTFT} W(\omega-\omega_0)$

$$\Rightarrow W_{sine}(\omega) = \frac{1}{2j} e^{j\frac{\pi}{2M}} \cdot W_{rect}\left(\omega - \frac{\pi}{M}\right) - \frac{1}{2j} e^{-j\frac{\pi}{2M}} W_{rect}\left(\omega + \frac{\pi}{M}\right)$$

$$= \frac{1}{2} e^{j\left(\frac{\pi}{2M}-\frac{\pi}{2}\right)} \cdot \frac{\sin\left(\frac{M}{2}\left(\omega-\frac{\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega-\frac{\pi}{M}\right)\right)} \cdot e^{-j\left(\frac{M-1}{2}\right)\cdot\left(\omega-\frac{\pi}{M}\right)}$$

$$+ \frac{1}{2} e^{j\left(-\frac{\pi}{2M}+\frac{\pi}{2}\right)} \frac{\sin\left(\frac{M}{2}\left(\omega+\frac{\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega+\frac{\pi}{M}\right)\right)} e^{-j\left(\frac{M-1}{2}\right)\left(\omega+\frac{\pi}{M}\right)}$$

$$= \frac{1}{2} e^{-j\frac{M-1}{2}\omega} \left\{ \frac{\sin\left(\frac{M}{2}\left(\omega-\frac{\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega-\frac{\pi}{M}\right)\right)} + \frac{\sin\left(\frac{M}{2}\left(\omega+\frac{\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega+\frac{\pi}{M}\right)\right)} \right\}$$

① $$W(\omega) = \text{DTFT}\{w[n]\} = \sum_{n=-\infty}^{\infty} w[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \{u[n] - u[n-M]\}\, e^{-j\omega n}$$

$$= \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1 - e^{j\omega M}}{1 - e^{j\omega}}$$

$$= \frac{e^{-j\omega\frac{M}{2}}\, \frac{1}{2j}\{e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}}\}}{e^{-j\omega\frac{1}{2}}\, \frac{1}{2j}\{e^{j\omega\frac{1}{2}} - e^{-j\omega\frac{1}{2}}\}}$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot \frac{\sin\frac{M}{2}\omega}{\sin\frac{1}{2}\omega}$$

② $$w[n] = \sin\left(\frac{\pi}{M}(n+0.5)\right)$$

$$= \sin\left(\frac{\pi}{M}n + \frac{\pi}{M}\cdot\frac{1}{2}\right)$$

$$= \sin\left(\frac{\pi}{M}n - \frac{\pi}{M}\cdot\frac{M-1}{2} + \frac{\pi}{M}\cdot\frac{M}{2}\right)$$

$$= \sin\left(\frac{\pi}{M}\left(n - \frac{M-1}{2}\right) + \frac{\pi}{2}\right)$$

$$= \cos\left(\frac{\pi}{M}\left(n - \frac{M-1}{2}\right)\right)$$

- Hanning & Hamming windows

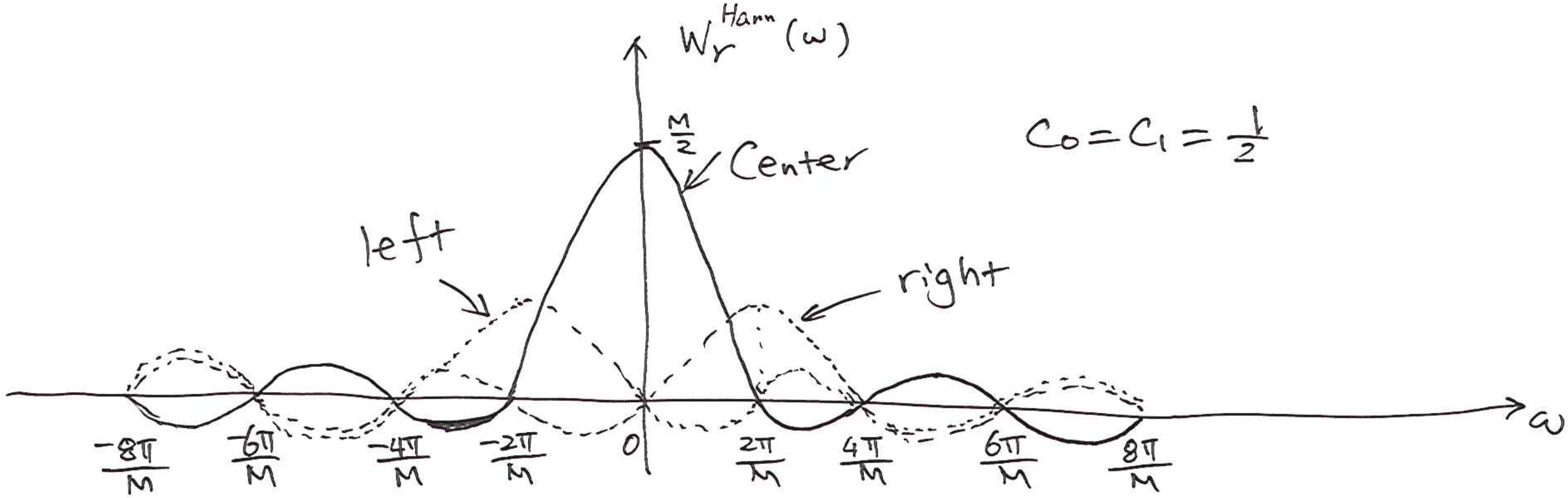
$$W[n] = C_0 - C_1 \cos\left(\frac{2\pi}{M}(n+0.5)\right) \qquad 0 \le n \le M-1$$

Hanning: $C_0 = C_1 = 0.5$ — low sidelobes and drops very fast

Hamming: $C_0 = .54, \; C_1 = .46$ — Equi-ripple, roughly $\sim -45\text{dB}$ down

$$W[n] = C_0 W_{rect}[n] - \frac{C_1}{2} e^{j\frac{\pi}{M}} e^{j\frac{2\pi}{M}n} W_{rect}[n] - \frac{C_1}{2} e^{-j\frac{\pi}{M}} e^{-j\frac{2\pi}{M}n} W_{rect}[n].$$

$$W(\omega) = e^{-j\frac{M-1}{2}\omega} \left\{ \frac{C_1}{2} \frac{\sin\left(\frac{M}{2}\left(\omega + \frac{2\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega + \frac{2\pi}{M}\right)\right)} + C_0 \frac{\sin\left(\frac{M}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} + \frac{C_1}{2} \frac{\sin\left(\frac{M}{2}\left(\omega - \frac{2\pi}{M}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - \frac{2\pi}{M}\right)\right)} \right\}$$



- Sidelobes of "left" and "right" are in-phase
- Sum together — sum patterns sidelobes 180° out-of-phase with "center" pattern
- Sum all three patterns to obtain reduced sidelobe levels
  - mainlobe peak $\frac{8\pi}{M}$
  - sidelobe : -32dB down