Sesssion 14

Outline

- Characteristics of some common analog filter design Techniques (8.3.5 in P&M)
- Bilinear Transformation Method of Design IIR Filters (Section 8.3.3 in P&M Text)

$$x_a(t) \rightarrow \boxed{\begin{matrix}\text{Analog filter}\\ h_a(t)\end{matrix}} \longrightarrow y_a(t)$$

$$\mathcal{L}\{h_a(t)\} = H_a(s) = \frac{Y_a(s)}{X_a(s)} = \frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - p_k)}$$

$$H_a(s) = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=1}^{N} \alpha_k s^k} \qquad \alpha_N = 1 .$$

(polynomial representation)

- $s = \sigma + j\Omega$ $\qquad$ Ω – analog angular frequency

$$f_a = \frac{\Omega}{2\pi} .$$

- Map from s-plane to z-plane $(z = re^{j\omega})$

$$H(z) = H_a(s)\Big|_{s = \frac{n(z)}{d(z)}} .$$

- Rational $H_a(s)$ leads to rational $H(z)$ which may be implemented as a difference equation.

- Mapping should also possess the following Properties:

(1) The $j\Omega$ axis in s-plane mapped onto (1-to-1) unit circle in z-plane

- guarantees equip-ripple (or monotonic-decreasing) property preserved through transformation

- note: 1-to-1 mapping between each F in $-\infty < F < +\infty$ and each ω in $-\pi < \omega < \pi$.

(2) LHP of s-plane mapped into inside of unit circle.

- guarantees poles in LHP of s-plane mapped into poles inside unit circle in z-plane
- Stable analog filter mapped to stable digital filter

- Bilinear Transformation:

$$s = c\,\frac{z-1}{z+1} \qquad c \text{ - constant}$$

this transformation ensures the preservance of the above properties.

- Investigate if mapping satisfies two desired properties:

$$s = \sigma + j\Omega \implies z = re^{j\omega}$$

$$\sigma + j\Omega = c\,\frac{re^{j\omega}-1}{re^{j\omega}+1}\left(\frac{re^{-j\omega}+1}{re^{-j\omega}+1}\right)$$

$$= c \cdot \frac{(r^2-1) + j2r\sin\omega}{r^2+2r\cos\omega+1}$$

- Equating real & imaginary parts on both sides of eqn.:

$$\sigma = c \, \frac{r^2 - 1}{r^2 + 2r\cos\omega + 1}$$

$$\Omega = c \, \frac{2r\sin\omega}{r^2 + 2r\cos\omega + 1}$$

- note: $\quad r^2 + 2r\cos\omega + 1 > 0$

- Observe:

$\sigma < 0$ (LHP) dictates:

$$r^2 - 1 < 0 \Rightarrow r^2 < 1 \Rightarrow |r| < 1.$$

- LHP s-plane mapped into inside of unit circle.

- Consider $\sigma = 0 \Rightarrow$ implies $\rightarrow r = 1$.

$\Rightarrow$ $j\Omega$-axis in s-plane is mapped to unit circle in z-plane.

$$s = re^{j\omega} = r\cos\omega + jr\sin\omega$$

$$s = \sigma + j\Omega \qquad \sigma = 0 \Rightarrow s = j\Omega$$

- Setting $r=1$ in 2^{nd} expression

$$\Rightarrow \Omega = c\,\frac{2\sin\omega}{2+2\cos\omega} = c\,\frac{\sin\omega}{1+\cos\omega}$$

$$\Rightarrow \Omega = c\,\frac{\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{\cos^2\frac{\omega}{2}} = c\tan\left(\frac{\omega}{2}\right)$$

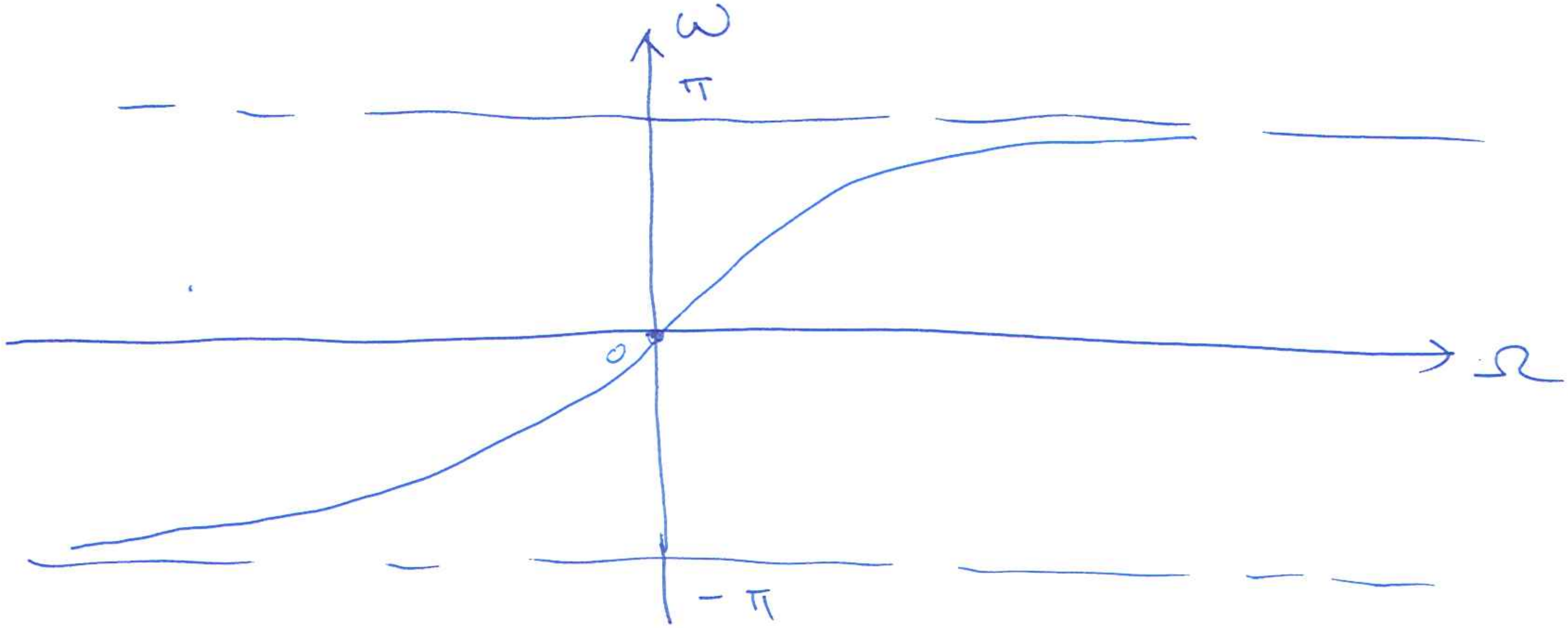
recall: $\cos^2\left(\frac{\omega}{2}\right) = \frac{1}{2}(1+\cos\omega)$

$\sin^2(c\omega) = 2\sin\frac{\omega}{2}\cos\frac{\omega}{2}$

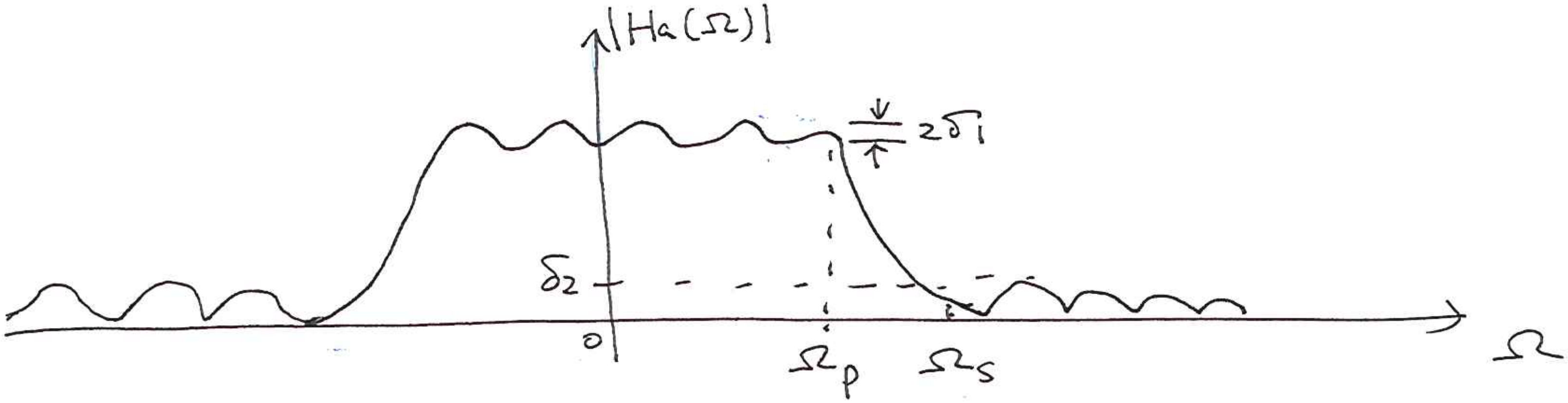
$$\Rightarrow \quad \boxed{\omega = 2\tan^{-1}\left(\frac{\Omega}{c}\right)}$$

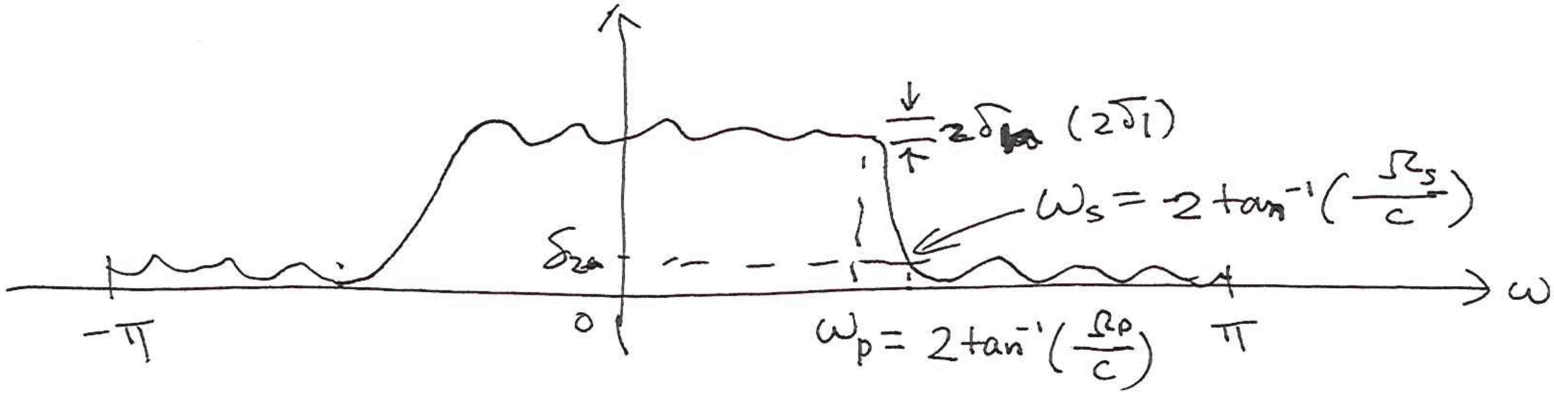
$\Omega \to \infty \quad \Rightarrow \quad \omega \to \pi$

$\Omega \to -\infty \quad \Rightarrow \quad \omega \to -\pi \qquad\qquad \Omega = 0 \Rightarrow \omega = 0.$



- 1-to-1 mapping as desired
- mapping is non-linear
- frequency compression (warping)





- Design Procedure:

1. Transform design specs in digital domain to analog domain: $\Omega_p = c\tan(\frac{\omega_p}{2})$, $\Omega_s = c\tan(\frac{\omega_s}{2})$

   $\delta_{1a} = \delta_1$, $\delta_{2a} = \delta_2$

2. design analog LPF to meet these spess (cookbook design — matlab does it for you).

3. $H(z) = H_a(s) \Big|_{s = c\frac{z-1}{z+1}}$

4. Convert to difference equation

- Simple illustrative example:
- design specs: single pole LPF with 3-dB cut-off at $\omega_c = 0.2\pi$.
- 1st order Butterworth filter: $H_a(s) = \Omega_c / s + \Omega_c$

$$\Omega_c = c \tan\left(\frac{0.2\pi}{2}\right) = 0.325c$$

$$H(z) = \frac{0.325c}{s + .325c}\Big|_{s = c\frac{z-1}{z+1}}$$

$$= \frac{0.325(z+1)}{z-1+0.325(z+1)} = \frac{0.245(1+z^{-1})}{1-0.509z^{-1}}$$

$$\Rightarrow y[n] = 0.509\, y[n-1] + 0.245\, x[n] + 0.245\, x[n-1].$$