Linear Programming

MGT 305

Instructor: George Mechling, Ph.D.

Linear Programming--Agenda

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- **Motivation**
- **Preliminaries--definitions, components, basic form**
- **Elementary example--set-up, graphical solution and sensitivity analysis**
- **Typical LP (linear programming) set-ups**
- **Computerized LP simplex solutions--interpretations**

Linear Programming--motivation

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- **Fundamental and Traditional Quantitative Method**
- **Applicability to making many different kinds of decisions**
- **Generates a wealth of information**
- **Robust method**

Linear Programming-preliminaries (1)

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Linear Programming (definition): **a *linear* mathematical technique for optimizing the achievement of a single objective that is subject to constraints on what can be done in achieving that objective. In LP, the *single objective* being optimized is in the form of a mathematical expression called the *objective function*. *Constraints* consist of a set mathematical expressions simply called *constraints* or the *constraint system*.**

Optimization (definition): **the act of maximizing or minimizing the value of some objective to the greatest extent possible.**

Components of a linear program:

The *objective function* (OF) consists generally of two or more variables called *decision variables*. The variables assume values about which decisions are made such that the value of the objective function, usually stated in monetary terms, is optimized. Associated with each of these values is a *coefficient* or multiplier, usually stated as a monetary rate. A *constraint* consists of at least one decision variable and an associated *coefficient* that is related to some constant called a *RHS value* by either a $\leq$, $\geq$, or =. The *coefficient* is a rate at which its decision variable approaches equaling the *RHS value*.

Linear Programming-preliminaries (2)

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Basic Form:

O.F. (max):

$$Z_{max} = c_{11}\cdot x_1 + c_{12}\cdot x_2 + c_{13}\cdot x_3 + \ldots + c_{1n}\cdot x_n$$

s.t. (constraints):

$$\begin{aligned} a_{11}\cdot x_1 + a_{12}\cdot x_2 + a_{13}\cdot x_3 + \ldots + a_{1n}\cdot x_n &\leq b_1 \\ a_{21}\cdot x_1 + a_{22}\cdot x_2 + a_{23}\cdot x_3 + \ldots + a_{2n}\cdot x_n &\leq b_2 \\ a_{31}\cdot x_1 + a_{32}\cdot x_2 + a_{33}\cdot x_3 + \ldots + a_{3n}\cdot x_n &\leq b_3 \\ &\vdots \\ a_{m1}\cdot x_1 + a_{m2}\cdot x_2 + a_{m3}\cdot x_3 + \ldots + a_{mn}\cdot x_n &\leq b_m \end{aligned}$$

decision variables

RHS constants

coefficients

NOTE: Constraints can be $\geq$ but at least one must be $\leq$. No restriction on # of = constraints except that all of them must intersect at the same point.

O.F. (min):

$$Z_{min} = c_{11}\cdot x_1 + c_{12}\cdot x_2 + c_{13}\cdot x_3 + \ldots + c_{1n}\cdot x_n$$

s.t.

$$\begin{aligned} a_{11}\cdot x_1 + a_{12}\cdot x_2 + a_{13}\cdot x_3 + \ldots + a_{1n}\cdot x_n &\geq b_1 \\ a_{21}\cdot x_1 + a_{22}\cdot x_2 + a_{23}\cdot x_3 + \ldots + a_{2n}\cdot x_n &\geq b_2 \\ a_{31}\cdot x_1 + a_{32}\cdot x_2 + a_{33}\cdot x_3 + \ldots + a_{3n}\cdot x_n &\geq b_3 \\ &\vdots \\ a_{m1}\cdot x_1 + a_{m2}\cdot x_2 + a_{m3}\cdot x_3 + \ldots + a_{mn}\cdot x_n &\geq b_m \end{aligned}$$

NOTE: Constraints can be $\leq$ but at least one must be $\geq$. No restriction on # of = constraints except that all of them must intersect at the same point.

Linear Programming-elementary example (1)

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Consider West LaPorte Lawn and Tree Specialists Service. The service consists of several crews of personnel cross-trained to perform both lawn-care and tree-work. Lawn care consists of cutting grass and edging and hedge trimming. Tree-work consists of pruning and crown thinning. Generally, it takes a crew 2 hours to do either job and the total number of crew-hours available is 240 hours/week. Given the equipment currently available to its crews, West LaPorte could perform as many as 160 lawn-care jobs/week. It can however, perform only half that many tree-work jobs in the same period of time. The lawn-care and tree-work service makes on average $30 profit for every lawn-care job and $45 for every tree-work job it performs. The service does not want for either kind of job for it to schedule and perform. How many jobs of either kind should be scheduled in a week's time so West LaPorte Lawn and Tree Specialists Service can maximize its profit for that week?

Setup

Let X_1 = # of lawn-care jobs and X_2 = # of tree-work jobs. Why????

Then the O.F. is: $Z_{max} = 30 \cdot X_1 + 45 \cdot X_2$ Why????

(labor constraint) $2 \cdot X_1 + 2 \cdot X_2 \leq 240$ **Why????**

(equipment capacity constraint) $1 \cdot X_1 + 2 \cdot X_2 \leq 160$ **Why????**

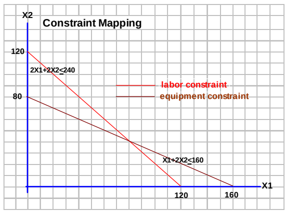
$X_1 , X_2 \geq 0$ **Why????**

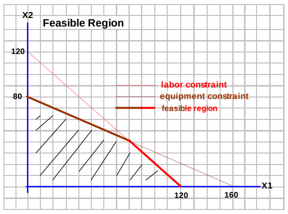
Graphical procedure **for solving the West LaPorte Lawn and Tree Specialists Service problem.**

- **Map the *constraint system* and establish the *feasibility region***
- **Map the O.F.**
- **Determine the point of tangency for the O.F. and feasibility region (the solution) and solution sensitivity**

Linear Programming-1st elementary example (2) 7

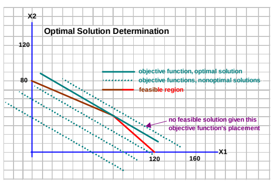
Graphical Analysis

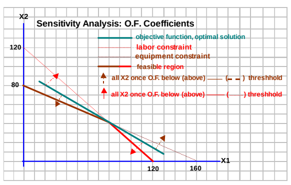




Linear Programming-1st elementary example (3) 8

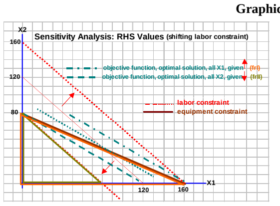
Graphical Analysis

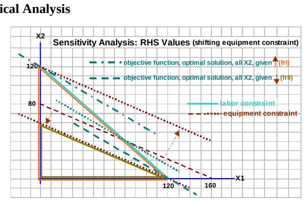




Linear Programming-1st elementary example (4) 9

Graphical Analysis





Linear Programming-2nd elementary example (5)

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Consider the Vice-Chancellor (VC) of South LaPorte University. The VC must plan the school's offerings for the fall semester. Student demands make it necessary to offer at least 30 undergraduate and 20 graduate courses in the term. Faculty contracts also dictate that at least 60 courses be offered in total. Each undergraduate course costs the the college an average of $2000 in faculty wages, and each graduate course costs $3500. How many undergraduates and undergraduate courses should be taught in the fall so that faculty salaries are kept to minimum?

Set-up

Let X_1 = # of undergraduate class offerings and X_2 = # of graduate class offerings. Why????

Then the O.F. is: $Z_{min} = 2000 \cdot X_1 + 3500 \cdot X_2$ **Why????**

(undergraduate constraint)	X_1		≥ 30	Why????
(graduate constraint)		X_2	≥ 20	Why????
(combined class offerings)	X_1	$+ X_2$	≥ 60	Why????
	$X_1, X_2 \geq 0$			Why????

Graphical procedure for solving the VC's South LaPorte University problem.

- **Map the *constraint system* and establish the *feasibility region***
- **Map the O.F.**
- **Determine the point of tangency for the O.F. and feasibility region (the solution)**

Linear Programming-2nd elementary example (6)

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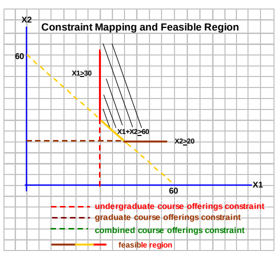

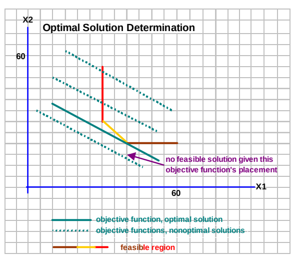

Linear Programming-Typical LP Set-ups (1)

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Performing graphical solutions of LP problems is a bit cumbersome and for LP problems consisting of 3 or more decision variables, solving them graphically may be regarded as virtually impossible. LP problems consisting of 3 or more decision variables are customarily solved using what is called the *Simplex* method. Using the simplex to solve even a 2 decision variable problem is however, a rather tedious affair. The ready availability of electronic computer power virtually eliminates this tedious technical task by doing in milli-seconds without mistakes what might take a person skilled in using the simplex several or many hours with some probability that mistakes were made. Exploiting this technological capability permits the introduction to many interesting, different, and often quite complex, yet as will be seen, easily solved LP problems.

While the ready availability of electronic computer power to solve these LP problems virtually eliminates the burdensome technical intricacies of solving them ourselves, there is one important and demanding task from which we are not relieved whether we had the computer or not. That task is the task of setting up LP problems. LP is a technique that is applied to many different kinds of business problems. *Fortunately*, there are prescribed ways of programming these problems. Thus, if you can recognize the kind of problem with which you are dealing, there is a way to set it up. Recognizing the kind of problem you are to program simplifies your task substantially and substantially reduces your risk of making errors. Some of the kinds of problems we can examine are:

Aggregate Planning (Product Mix)
Resource Allocation
Ingredients Mix (Diet Problem)
Blend
Scheduling
Multi-period Investment
Workforce Planning
Transportation

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Linear Programming-Typical LP Set-ups (2)

Aggregate Planning (Product Mix): This problem typically involves determining what combination of outputs will maximize an O.F. for a given set of ($\leq$) resource constraints. ($\geq$ or = constraints may also be present but at least one $\leq$ must always be present.) Usually, the O.F. is some measure of profit. Here, the decision variables stand for the different outputs. The value a decision variable one assumes in the solution to this kind of problem is a count of the number of units of a given output for which that variable stands. The West LaPorte Lawn and Tree Service problem is an example of this kind of problem. *Remember*, this problem is about maximizing output for given set of resources. So the constraints involve rates at which outputs expend resources.

Resource Allocation (Input-Mix): This problem is the reverse of the aggregate planning problem. It involves determining the combination of resources (in the O.F.) that must be expended to achieve a given set of outputs (the constraint system). *Remember*, this problem is about *minimizing the O.F.*--the combined cost of the resources needed to generate a given level of outputs. The decision variables stand for some measure of resource usage and the $\geq$ constraints involve rates at which resources generate outputs. ($\leq$ or = constraints may also be present but at least one $\geq$ must always be present.) The South LaPorte University example was this kind of problem.

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Linear Programming-Typical LP Set-ups (3)

Ingredients Mix (Diet Problem): This problem is similar to the resource allocation problem except that it specifically involves minimizing the cost of meeting given nutritional requirements by determining what should be the optimal combination from an available set of foodstuffs for, e.g., a meal . In this problem, the decision variables stand for the different kinds of foodstuffs. Cost coefficients are associated with these variables in the O.F. which is minimized. The $\geq$ constraints involve rates at which units of these foodstuffs generate the set of nutritional requirements stipulated. ($\leq$ or = constraints may also be present but at least one $\geq$ must always be present.)

Your farm produce shop has an order to mix up a bag of feed for one of your customers. This feed mix is made from two different ingredients, A and B. Ingredient A contains 1 lb. of vitamins and 4 lbs. of minerals that are required to be present in the feed mix. Ingredient B contains 2 lbs. of vitamins and 2 lbs. of minerals that are required to be present in the feed mix. The feed mix must contain at least 40 lbs. of the required vitamins and 60 lbs. of the required minerals. Because of additional nutrients in the ingredients, the feed mix must contain at least five bags of each ingredient. A bag of ingredient A costs $8 and a bag of ingredient B costs $10. How many bags of each ingredient should you, the shop owner, use to minimize your costs?

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Linear Programming-Typical LP Set-ups (4)

Blend: This problem is similar to the ingredients mix problem inasmuch as it involves mixing ingredients in such a way as to conform to certain standards set forth in the constraint system. This problem however, differs in a number of significant ways. First, an most importantly, there is more than one product being "mixed up." That is, 3 ingredients may be mixed in different ways to generate 2, 3, or more products. If e.g., 3 products are to be generated from 3 ingredients mixed in different ways, the O.F. must consist of 9 decision variables--3 decision variables standing for the 3 ingredients in one product, 3 more decision variables standing for the same 3 ingredients in the next product, etc. The O.F. is most always maximized with its coefficients representing profits achieved from the use of the various ingredients in the various products. One should be alert *however*, that there is no reason to believe that a cost minimization in the form of a blend problem could not be devised.

Your farm produce shop has orders for 5,000 lbs. of fertilizer A and 3,000 lbs. of fertilizer B. You mix up these two fertilizers using two components (C_1 and C_2) containing nitrogen and phosphorus. C_1 costs $0.60/lb. and contains 50% nitrogen and 20% phosphorus by weight. C_2 costs $0.50/lb. and contains 20% nitrogen and 40% phosphorus by weight. Fertilizer A must contain at least 30% nitrogen and 20% phosphorus and fertilizer B must contain at least 20% nitrogen and 40% phosphorus. How many pounds of each ingredient should you mix for each type of fertilizer to minimize your costs?

Linear Programming-Typical LP Set-ups (5)

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Work-shift scheduling: This problem involves the scheduling of shifts of varying numbers of employees for a specified duration of time at specified times to meet the demand for their services in a timely fashion. Most often the duty times of these shifts will overlap, thus complicating the short-term resource planning of the decision-maker. Generally, it is the case To minimize the number of employees used to satisfy the demand for their services in terms of their physical count or the cost of employing them.

The owner of a hamburger stand operates his stand 24 hrs/day. His shifts report for work one every 4 hrs. and each works 8 hrs. The owner figures that from midnight to 4 a.m., 3 employees are needed. From 4 a.m. to 8 a.m., 5 are needed. From 8 a.m. to 12 noon, 13 are needed. From 12 noon to 4 p.m., 8 are needed. From 4 p.m. to 8 p.m., 19 are needed. And from 8 p.m. to 12 midnite, 10 are needed. Construct an LP model of this problem that will minimize the number of employees the owner needs for his stand while satisfying the estimated need for services

Linear Programming-Typical LP Set-ups (6)

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Multiperiod investment: This problem generally involves a given amount of money that is to be invested in two or more different kinds of interest bearing instruments over a period of time consisting of 2, 3, 4 or more time periods. Any one or all of these different kinds of interest bearing Investment instruments may require more than one of these “time periods” to mature. Furthermore, the investing in these instruments that takes place is subject to portfolio constraints that contribute toward defining the investment package. The objective of the problem is usually that of maximizing the return of the money invested by the close of the 2, 3, 4 or more time period investment period.

A firm has $1M to invest in stocks, bonds, CDs and real estate. The firm wishes to maximize the value of its cash at the end of 6 years. Stocks mature in 2 years and provide a return at that time of 20%. Bonds mature in 3 years and provide a return at that time of 40%. CDs mature in 4 years, provide a return of 80% at that time, and are available only once, at the beginning of the second year. Real estate investments are available at the beginning of the 5th and 6th year and provided yearly return of 10%. In order to minimize risk, the firm has decided to diversify its investments. Investments in stocks cannot exceed 30% of total investments and at least 25% of the total investment must be in CDs.

Linear Programming-Typical LP Set-ups (7)

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Workforce planning (example 17.5 from text): This problem involves the hiring, training, inevitable turnover of an organization’s workforce, and providing for the staffing requirement to do this. Planning decisions of this sort must be informed by anticipated changes in consumer demand. Such activity to maintain as well as expand an organizations workforce has costs. Therefore, part of the decision-making process devising these plans necessarily involves implementing in the least-cost way.

A bank must decide how many new tellers to hire and train over the next 6 months. Required teller-hours needed/month are 1500 (Jan), 1800 (Feb), 1600 (Mar), 2000 (Apr), 1800 (May), and 2200 (Jun). A new teller requires one month of training and 80 hours of simulated job experience during that month supervised by an experienced teller before assuming duty. Experienced tellers not supervising simulation training work 160 hrs each month. The bank has 12 experienced tellers at the beginning of January. Turnover of experienced tellers is 10% per month. Experienced tellers are paid $600/month and trainees are paid $300/month. Construct a linear program that will satisfy the staffing, hiring, and training needs of the the bank while doing so at the minimum cost.