Involutory Decomposition of Groups Into Twisted Subgroups and Subgroups

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Abstract. An involutory decomposition is a decomposition, due to an involution, of a group into a twisted subgroup and a subgroup. We study unexpected links between twisted subgroups and gyrogroups. Twisted subgroups arise in the study of problems in computational complexity. In contrast, gyrogroups are grouplike structures which first arose in the study of Einstein's velocity addition in the special theory of relativity. Particularly, we show that every gyrogroup is a twisted subgroup and that, under general specified conditions, twisted subgroups are gyrocommutative gyrogroups. Moreover, we show that gyrogroups abound in group theory and that they possess rich structure.

§1. Introduction

Under general conditions, twisted subgroups are near subgroups [1]. Feder and Vardi [4] introduced the concept of a near subgroup of a finite group as a tool to study problems in computational complexity involving the class NP. Aschbacher provided a conceptual base for studying near subgroups demonstrating that near subgroups possess much structure, so that it seems likely that [1] "one can completely describe all near subgroups [of finite groups] in some sense, using classification of the finite simple groups."

Gyrogroups are essentially equivalent to gyrotransversals which are twisted subgroups, Theorem 3.8. Gyrogroups are special loops which share remarkable analogies with groups. The first known gyrogroup structure is the relativistic gyrogroup $(\Re^3_1, \oplus)$ that appeared in 1988 [8], consisting of the unit ball $\Re^3_1$ of the Euclidean 3-space $\Re^3_1$ with Einstein's addition. The Einstein velocity addition $\oplus$ of relativistically admissible velocities is a binary operation in the unit ball $\Re^3_1 = \{x \in \Re^3 : ||x|| < 1\}$ of the Euclidean 3-space $\Re^3$, where the vacuum speed of light is normalized to $c = 1$. Counterintuitively, the Einstein velocity addition is neither commutative nor associative. Is the progress from the common vector addition of velocities $+$, which is a group operation, to the Einstein velocity addition $\oplus$, which is not a group operation, associated with loss of mathematical regularity?

It has been shown in [11] that the group structure that has been lost in the transition from the group $(\Re^3, +)$ to the nongroup groupoid $(\Re^3_1, \oplus)$ is replaced by a loop structure using a relativistic peculiar rotation called the *Thomas precession.* Extending the Einstein relativistic groupoid $(\Re^3_1, \oplus)$ with its Thomas precession by abstraction, the gyrogroup notion emerges, where the abstract Thomas precession is called the *Thomas gyration.* The Thomas gyration has its own life due to powerful properties that it possesses. It suggests the prefix *gyro* that we extensively use to emphasize analogies. Thus, for instance, gyrogroup operations are gyroassociative and, sometimes gyrocommutative, in full analogy with group operations which are associative and, sometimes commutative. Moreover, some commutative groups can be extended to vector spaces by the introduction of scalar multiplication and inner product and, in full analogy, some gyrocommutative gyrogroups can be extended to gyrovector spaces. Then, unexpectedly, gyrovector spaces provide the setting for hyperbolic geometry in full analogy with vector spaces, that provide the setting for Euclidean geometry [11,13]. Hence, the hyperbolic geometry of Bolyai-Lobachevski is in fact the gyro-Euclidean geometry. Like groups, there are finite and infinite gyrogroups some of which are gyrocommutative. We obtain particularly interesting results when the order of the group in which a gyrogroup resides as a twisted subgroup is odd.

Unexpectedly, gyrogroups are twisted subgroups and, under specified general conditions, twisted subgroups are gyrogroups. By demonstrating the rich structure of gyrogroups, and by demonstrating that gyrogroups and twisted subgroups are intimately related, we support Aschbacher's observation that [1] "near subgroups possess much structure." Furthermore, Aschbacher's hope to completely describe all near subgroups in some sense, using the classification of the finite simple groups, may result in the complete description of all finite gyrogroups as well.

To show that gyrogroups abound in group theory we show in Theorem 5.1 that every odd order group that possesses an involutory automorphism contains a gyrocommutative gyrogroup. More specifically, we find that any involutory automorphism of an odd order group gives rise to a unique decomposition, called the *involutory decomposition*, that decomposes the group into the product of a twisted subgroup and a subgroup. We identify the twisted subgroup factor in the involutory decomposition as a gyrocommutative gyrogroup. Gyrogroups are, thus, everywhere in group theory, lying dormant waiting for their discovery. The gyrogroup definition follows.

§2. Gyrogroups, and Groups Containing Gyrogroups

Definition 2.1. (Groupoids or Magmas, and Automorphism Groups of Groupoids) A groupoid [3] or a magma [2] is a nonempty set with a binary operation. An automorphism of the groupoid $(S, \cdot)$ is a bijection of S that respects the binary operation $\cdot$ in S. The set of all automorphisms of $(S, \cdot)$ forms a group denoted by $Aut(S, \cdot)$.

An important subcategory of the category of groupoids is the category of loops, which are defined below.

Definition 2.2. (Loops [3]) A *loop* is a magma $(S, \cdot)$ with an identity element in which each of the two equations $a \cdot x = b$ and $y \cdot a = b$ for the unknowns x and y possesses a unique solution ($a \cdot x$ is the product of a and x in S. Subsequently we will omit the dot and write ax).

Being nonassociative, the Einstein velocity addition of relativistically admissible velocities in the special theory of relativity is not a group operation. A gyrogroup is a special grouplike loop that has been abstracted from the groupoid of all relativistically admissible velocities with their Einstein's velocity addition and Thomas' precession [11]. The abstract Thomas precession is called the *Thomas gyration*, suggesting the prefix *gyro* that we extensively use to emphasize analogies.

Definition 2.3. (Gyrogroups [11], Left Gyrogroups) The magma $(G, \odot)$ is a *gyrogroup* if its binary operation satisfies the following axioms. In G there is at least one element, 1, called a left identity, satisfying

(G1) $1 \odot a = a$ Left Identity

for all $a \in G$. There is an element $1 \in G$ satisfying axiom $(G1)$ such that for each a in G there is an x in G, called a left inverse of a, satisfying

(G2) $x \odot a = 1$ Left Inverse

Moreover, for any $a, b, z \in G$ there exists a unique element $gyr[a, b]z \in G$ such that

(G3) $a \odot (b \odot z) = (a \odot b) \odot gyr[a, b]z$ Left Gyroassociative Law

If $gyr[a, b]$ denotes the map $gyr[a, b] : G \to G$ given by $z \mapsto gyr[a, b]z$ then

(G4) $gyr[a, b] \in Aut(G, \odot)$ Gyroautomorphism

and $gyr[a, b]$ is called the *Thomas gyration*, or the *gyroautomorphism* of G, *generated* by $a, b \in G$. Finally, the gyroautomorphism $gyr[a, b]$ generated by any $a, b \in G$ satisfies

(G5) $gyr[a, b] = gyr[a \odot b, b]$ Left Loop Property

A magma $(G, \odot)$ is a *left gyrogroup* if it satisfies axioms (G1)-(G4), and

(G5') $gyr[a, a^{-1}] = id$ Weak Loop Property

Definition 2.4. (Gyrocommutative Gyrogroups) The gyrogroup $(G, \oplus)$ is *gyrocommutative* if for all $a, b \in G$

(G6) $a \oplus b = gyr[a, b](b \oplus a)$ Gyrocommutative Law

As it is customary with groups, we use additive notation, $\oplus$, with gyrocommutative gyrogroups, and multiplicative notation, $\odot$, with general gyrogroups.

Definition 2.5. (Gyrations, Gyroautomorphisms, Gyroautomorphism Groups) The automorphisms $gyr[a, b]$ of a gyrogroup are called *gyroautomorphisms*. The action of the gyroautomorphism $gyr[a, b]$ on G is called a gyration. The set of all gyroautomorphisms of a gyrogroup $(G, \odot)$ need not form a group. A *gyroautomorphism group* of $(G, \odot)$ is any subgroup $Aut_o(G, \odot)$ (not necessarily the smallest one) of $Aut(G, \odot)$ containing all the gyroautomorphisms of $(G, \odot)$.

Properties of gyrogroups have been studied in [11] where, in particular, the following alternative, equivalent definition of a gyrogroup is presented.

Theorem 2.6 (Gyrogroups - an Alternative Definition [11]**).** *A magma $(G, \odot)$ is a gyrogroup if its binary operation satisfies the following axioms and properties. In G there exists a unique element, 1, called the identity, satisfying*

(g1) $1 \odot a = a \odot 1 = a$ *Identity*

for all $a \in G$. For any $a \in G$ there exists in G a unique inverse, a^{-1}, satisfying

(g2) $a^{-1} \odot a = a \odot a^{-1} = 1$ *Inverse*

Moreover, if for any $a, b \in G$ the map $gyr[a, b]$ of G into itself is given by the equation

$$gyr[a, b]z = (a \odot b)^{-1} \odot (a \odot (b \odot z))$$

for all $z \in G$, then the following hold for all $a, b, c \in G$.

(g3) $gyr[a, b] \in Aut(G, \odot)$ *Gyroautomorphism*

(g4a) $a \odot (b \odot c) = (a \odot b) \odot gyr[a, b]c$ *Left gyroassociative Law*

(g4b) $(a \odot b) \odot c = a \odot (b \odot gyr[b, a]c)$ *Right gyroassociative Law*

(g5a) $gyr[a, b] = gyr[a \odot b, b]$ *Left Loop Property*

(g5b) $gyr[a, b] = gyr[a, b \odot a]$ *Right Loop Property*

Twisted subgroups prove useful as a tool to study problems in computational complexity [1]. We will see in this article that gyrogroups are intimately related to twisted subgroups, which are defined below.

Definition 2.7. (Twisted Subgroups [1]) A subset P of a group G is a twisted subgroup of G if (i) $1_G \in P$, 1_G being the identity element of G; and (ii) $aPa \subseteq P$ for all $a \in P$.

Every gyrogroup is a twisted subgroup of some specified group (Theorems 2.12 and 3.8), and some twisted subgroups are gyrogroups (Corollary 3.11). To expose the relationship between gyrogroups and twisted subgroups we will explore properties of gyrogroups in terms of groups that contain them as a subset. To enable us to study gyrogroups in terms of groups that may contain them we introduce the following definitions and a theorem.

Definition 2.8. (Transversals, the Transversal Operation, the Transversal Map, and Transversal Groupoids) A set B is a transversal in a group G (all transversals in this article are left transversals) of a subgroup H of G if every $g \in G$ can be written uniquely as $g = bh$ where $b \in B$ and $h \in H$. Let $b_1, b_2 \in B$ be any two elements of B, and let

$$b_1 b_2 = (b_1 \odot b_2)h(b_1, b_2) \tag{2.1}$$

be the unique decomposition of the element $b_1 b_2 \in G$, where $b_1 \odot b_2 \in B$ and $h(b_1, b_2) \in H$, determining (i) a binary operation, $\odot$, in B, called the *transversal operation of B induced by G,* and (ii) a map $h : B \times B \to H$, called the *transversal map.* The element $h(b_1, b_2) \in H$ is called the element of H determined by the two elements b_1 and b_2 of its transversal B in G (its importance stems from the fact that it gives rise to gyrations in Definition 2.10 below. Gyrations, in turn, result from the abstraction of the Thomas precession of the special theory relativity into the Thomas gyration). A *transversal groupoid* $(B, \odot)$ of H in G is a groupoid formed by a transversal B of H in G with its transversal operation $\odot$.

Definition 2.9. (Gyrotransversals, Gyrotransversal Groupoids, Gyro -Decompositions of Groups) A transversal groupoid $(B, \odot)$ of a subgroup H in a group G is a *gyrotransversal* of H in G if

(i) $1_G \in B$, 1_G being the identity element of G;

(ii) $B = B^{-1}$; and

(iii) B is normalized by H, $H \subseteq N_G(B)$, that is, $hBh^{-1} \subseteq B$ for all $h \in H$.

A *gyrotransversal groupoid* is a groupoid formed by a gyrotransversal with its transversal operation. The decomposition $G = BH$ where $H < G$ and where B is a transversal of H in G is a *gyro-decomposition* if B is a gyrotransversal of H in G. The gyro-decomposition $G = BH$ is *reduced* if $C_H(B) = \{1_G\}$.

Notation. In this paper we will use the notation $b^h = hbh^{-1}$ as in [5].

Definition 2.10. (Gyrations of a Gyrotransversal) Let B be a gyrotransversal of a subgroup H in a group $G = BH$, let $b_1, b_2 \in B$ be any two elements of B, and let $h(b_1, b_2)$ be the element of H determined by b_1 and b_2, h being the transversal map $h : B \times B \to H$, Def. 2.8. Then the gyration $gyr[b_1, b_2]$ of B generated by b_1 and b_2 is the map of B into itself given by

$$gyr[b_1, b_2] = \alpha_{h(b_1, b_2)}$$

where α_h, $h \in H$, denotes conjugation by h, that is, for any $h \in H$ and $b \in B$

$$\alpha_h(b) = b^h = hbh^{-1}$$

It follows from Definition 2.10 that a gyration of B generated by $b_1, b_2 \in B$ is given in terms of its effects on $x \in B$ by the equation

$$gyr[b_1, b_2]x = h(b_1, b_2)x(h(b_1, b_2))^{-1} \tag{2.2a}$$

or, equivalently, by the equation

$$gyr[b_1, b_2]x = x^{h(b_1,b_2)} \tag{2.2b}$$

The conjugation operations α_h, $h \in H$, are bijections of B since B is normalized by H. Hence, in particular, the gyrations $gyr[b_1, b_2]$ are bijections of B for all $b_1, b_2 \in B$. Moreover, the gyrations of a gyrotransversal B are automorphisms of the gyrotransversal groupoid $(B, \odot)$ as shown in the following

Theorem 2.11. *Let $(B, \odot)$ be a gyrotransversal groupoid of a subgroup H in a group G. Then, for any $b_1, b_2 \in B$*

$$gyr[b_1, b_2] \in Aut(B, \odot)$$

Proof. Since, by Eq. (2.2b), $gyr[a, b]x = x^{h(a,b)}$ for all $x \in B$, we have to show that

$$(x \odot y)^{h(a,b)} = x^{h(a,b)} \odot y^{h(a,b)}$$

for all $a, b, x, y \in B$.

More generally, however, we will verify the desired identity for any $k \in H$ regardless of whether or not k possesses the form $k = h(a, b)$. We will thus show that

$$(x \odot y)^k = x^k \odot y^k$$

for any $k \in H$. Clearly, we have in G

$$(xy)^k = x^k y^k \tag{2.3}$$

Employing the unique decomposition $G = BH$, Eq. (2.1), for both sides of (2.3) we have

$$(xy)^k = ((x \odot y)h(x, y))^k = (x \odot y)^k h(x, y)^k \tag{2.4}$$

on one hand, and

$$x^k y^k = (x^k \odot y^k)h(x^k, y^k) \tag{2.5}$$

on the other hand. It follows from (2.3) - (2.5) and from the uniqueness of the decomposition $G = BH$ that

$$(x \odot y)^k = x^k \odot y^k \tag{2.6}$$

and, by the way,

$$h(x^k, y^k) = h(x, y)^k \tag{2.7}$$

Eq. (2.6) completes the proof. $\square$

Theorem 2.12. *(Representation Theorem for Gyrogroups) If $(B, \odot)$ is a left-gyrogroup and H a gyroautomophism group of $(B, \odot)$, then there is a group G in which H is a subgroup of G and $(B, \odot)$ is a gyrotransversal groupoid of H such that for each $h \in H$ and $x \in B$, $h(x) = x^h$.*

Proof. This result is proved in section 4 of [11] for gyrogroups, but the proof does not use the left loop property so that it is valid for left gyrogroups. □

Theorem 2.13. *Any gyrotransversal B of a subgroup H in G is a left gyrogroup*

Proof. We have to show that $(B, \odot)$ satisfies axiomas (G1)-(G4). Axiom (G4) is verified in Theorem 2.11. It therefore remains to establish the validity of axioms (G1)-(G3).

Given $b \in B$ we get

$$b = 1b = (1 \odot b)h(1, b) \quad \text{and} \quad 1 = b^{-1}b = (b^{-1} \odot b)h(b^{-1}, b)$$

Hence,(G1) and (G2) are verified from the uniqueness of the decomposition.

for all $a, b, c \in B$ we clearly have in G

$$(ab)c = a(bc) \tag{2.8}$$

Employing the uniqueness of the decomposition for both sides of (2.8) we have

$$\begin{aligned} (ab)c &= (a \odot b)h(a, b)c \\ &= (a \odot b)gyr[a, b]ch(a, b) \\ &= ((a \odot b) \odot gyr[a, b]c)h(a \odot b, gyr[a, b]c)h(a, b) \end{aligned} \tag{2.9}$$

on one hand, and

$$\begin{aligned} a(bc) &= a(b \odot c)h(b, c) \\ &= (a \odot (b \odot c))h(a, b \odot c)h(b, c) \end{aligned} \tag{2.10}$$

on the oher hand. It follows from (2.8)-(2.10) and from the uniqueness of the decomposition that

$$(a \odot b)gyr[a, b]c = a \odot (b \odot c)$$

thus verifying (G3). □

§3. Gyrotransversals are Twisted Subgroups

We will show in this section that a gyrotransversal groupoid in a group is a gyrogroup if and only if the gyrotransversal is a twisted subgroup of G when the group G containing the twisted subgroup is *reduced* in the sense of Definition 3.4 below. In the following Lemma, $C_H(B)$ denotes the centralizer of B in $H < G$.

Lemma 3.1. *Let $G = BH$ be a group where $H < G$. If $H \subseteq N_G(B)$ then $C_H(B) \triangleleft G$.*

Proof. Let $A = \langle B \rangle$. Since $G = BH$, we have $G = AH$ and since H acts on B, H acts on A, so $C_H(B) = C_H(A) \trianglelefteq H$. Finally A acts on $C_H(A)$, so $G = AH \leq N_G(C_H(A))$. □

Lemma 3.2. *Let $(B, \odot)$ be the gyrotransversal groupoid of a subgroup H of a group G, $G = BH$. Then $(B, \odot)$ is the gyrotransversal groupoid of the subgroup $H_B = H/C_H(B)$ in the group $G_B = G/C_H(B)$.*

Proof. The permutation representation of G on G/H is equivalent to the representation of G on B by conjugation. The group G_B is just the image of G under this representation. □

Definition 3.3. (Transversal Enveloping Pairs) Let B be a transversal of a subgroup H in a group G. We say that G (H) is an enveloping group (subgroup) of the transversal B. Furthermore, we say that (G, H) is an enveloping pair of the transversal B and of the transversal groupoid $(B, \odot)$.

Definition 3.4. (Reduced Enveloping Pairs of Gyrotransversals) Let B be a gyrotransversal with an enveloping pair (G, H). The corresponding reduced enveloping pair of the gyrotransversal B and of the gyrotransversal groupoid $(B, \odot)$ is the pair

$$(G_B, H_B) = (G/C_H(B), H/C_H(B))$$

It follows from Lemma 3.2 that a reduced enveloping pair of a gyrotransversal groupoid is an enveloping pair of the gyrotransversal groupoid.

We clearly have the following Lemma, which exposes the importance of reducing enveloping pairs.

Lemma 3.5. *If the enveloping pair (G, H) of the gyrotransversal B of H in G is reduced, that is, $(G, H) = (G_B, H_B)$, then $C_H(B) = \{1_H\}$ is the trivial group consisting of the identity element of H.*

Theorem 3.6. *Let B be a gyrotransversal with a reduced enveloping pair (G, H). Then the map $h \mapsto \alpha_h$ is a bijection of $h(B \times B)$ with the set of all gyrations of B.*

Proof. By definition, the gyrations $gyr[b_1, b_2]$ of B, $b_1, b_2 \in B$ correspond to elements of $h(B \times B) \subseteq H$ by the relation

$$gyr[b_1, b_2] = \alpha_{h(b_1, b_2)}$$

where α_h is the inner automorphism of B given by $\alpha_h b = b^h$ for all $b \in B$ and $h \in H$. And the map $\alpha : h \mapsto \alpha_h$ is an injective group homomorphism from H into the symmetric group on B as $\ker(\alpha) = C_H(B) = \{1_H\}$, so that its restriction to $h(B \times B)$ is injective. □

It follows from Lemma 3.2 and Definition 3.3 that in the study of gyrotransversal groupoids on their own merits, rather than on merits of the group where they reside, one may assume without loss of generality that any gyrotransversal under consideration resides in one of its reduced enveloping groups. This results in the advantage of having a bijective correspondence between gyrations and transversal maps. Specifically, let $(B, \odot)$ be a gyrotransversal groupoid with a reduced enveloping pair (G, H). Then, there exists a bijective correspondence between the gyrations $gyr[b_1, b_2] \in Aut(B, \odot)$ of the gyrotransversal B and the elements $h(b_1, b_2) \in H$ of the image in H of the transversal map h. As an example, we present the reduced enveloping pair of Einstein's gyrogroup $(\Re^n_c, \oplus_E)$, where $\Re^n_c = \{v \in \Re^n : ||v|| < c\}$ is the open c-ball of the Euclidean n-space $\Re^n$, and where $\oplus_E$ is the Einstein addition, defined in [11] (No explicit presentation of $\oplus_E$ is needed in Example 3.7).

Example 3.7. The Lorentz group, parametrized by a velocity and an orientation parameter, is a group of pairs

$$L = \{(\mathbf{v}, V) : \mathbf{v} \in \Re^n_c,\ V \in SO(n)\} \tag{3.1a}$$

with group operation given by

$$(\mathbf{u}, U)(\mathbf{v}, V) = (\mathbf{u} \oplus_E U\mathbf{v}, gyr[\mathbf{u}, U\mathbf{v}]UV) \tag{3.1b}$$

where $\oplus_E$ is the Einstein velocity addition [11] and where $SO(n)$ is the special orthogonal group. The Lorentz group L and the orthagonal group $SO(n)$ constitute a reduced enveloping pair, $(L, SO(n))$, of the Einstein gyrotransversal $(\Re^n_c, \oplus_E)$.

We are now in a position to state the conditions under which twisted subgroups and gyrogroups are equivalent.

Theorem 3.8. *A gyrotransversal groupoid* $(P, \odot)$ *with a reduced enveloping pair* (G, H) *is a gyrogroup if and only if* P *is a twisted subgroup of* G*, and* $h(a, b) = h^{-1}(b, a)$ *(note from the proof that* $h(a, b) = h^{-1}(b, a)$ *is satisfied in the gyrocommutative case).*

Proof. Let us assume that P is a twisted subgroup. Then, $aba \in P$ for any $a, b \in P$. We wish to show that the gyrotransversal groupoid $(P, \odot)$ of H in G is a gyrogroup. Clearly, $bb = b1_G b \in P$. Hence, $abba \in P$. But, in G for the gyrocommutative case,

$$\begin{aligned} abba &= (ab)(ba) \\ &= (a \odot b)h(a, b)(b \odot a)h(b, a) \\ &= (a \odot b)(b \odot a)^{h(a,b)}h(a, b)h(b, a) \end{aligned}$$

implying

$$h(a, b)h(b, a) = 1_H$$

Hence,

$$h^{-1}(b, a) = h(a, b)$$

Similarly, since $aba \in P$ we have in G

$$\begin{aligned} aba &= a(ba) \\ &= a(b \odot a)h(b, a) \\ &= a \odot (b \odot a)h(a, b \odot a)h(b, a) \end{aligned}$$

implying

$$h(a, b \odot a)h(b, a) = 1_H$$

so that

$$h(a, b \odot a) = h^{-1}(b, a) = h(a, b)$$

thus obtaining the right loop property $h(a, b \odot a) = h(a, b)$. Inverting by means of $h^{-1}(a, b) = h(b, a)$ we obtain the desired left loop property for h, $h(b \odot a, a) = h(b, a)$ for all $a, b \in P$.

Since $gyr[a, b] = \alpha_{h(a,b)}$ we have

$$gyr[a \odot b, b] = gyr[a, b]$$

for all $a, b \in P$. Hence, the gyrotransversal groupoid $(P, \odot)$ possesses the left loop property. By Theorem 2.13 the gyrotransversal groupoid $(P, \odot)$ is therefore a gyrogroup.

Conversely, we now assume that $(P, \odot)$ is a gyrogroup. Let $a, b \in P$ be any two elements of P. We will show that the composition aba in G is an element of P. Gyrations $gyr[a, b]$ in P are in bijective correspondence with elements $h(a, b) \in H$ of the image $h(B \times B)$ of h in H, by Theorem 3.6. Hence, the (left and) right loop property for $gyr[a, b]$ is valid for $h(a, b)$ as well, that is, $gyr[a, b] = gyr[a, b \odot a]$, implying $h(a, b) = h(a, b \odot a)$. Similarly, the identity $gyr^{-1}[a, b] = gyr[b, a]$ implies $h^{-1}(a, b) = h(b, a)$. Following these properties of h we have in G

$$\begin{aligned} aba &= a(ba) \\ &= a(b \odot a)h(b, a) \\ &= a \odot (b \odot a)h(a, b \odot a)h(b, a) \\ &= a \odot (b \odot a)h(a, b)h(b, a) \\ &= a \odot (b \odot a) \in P \end{aligned}$$

for all $a, b \in P$. Hence, P is a twisted subgroup of G, thus completing the proof. □

Corollary 3.9. *Let $G = AH$, $H < G$, be a gyro-decomposition of G, Def. 2.9. If A is a twisted subgroup of G then A is a gyrogroup.*

Proof. The proof follows from the first part of the proof of Theorem 3.8. □

Corollary 3.9 suggests the following definition.

Definition 3.10. (Gyro-Twisted Subgroups) A twisted subgroup P in a group G is a *gyro-twisted subgroup* if P is a gyrotransversal of some subgroup $H < G$ in G.

Following Definition 3.10, Corollary 3.9 can now be stated as

Corollary 3.11. *Any gyro-twisted subgroup P in a group G is a gyrogroup $(P, \odot)$, whose gyrogroup operation is the transversal operation of P induced by G.*

Example 3.12. The most general Möbius transformation of the complex unit disc $D = \{z : |z| < 1\}$ in the complex z-plane [12],

$$z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \oplus z)$$

defines the Möbius addition $\oplus$ in the disc, allowing the Möbius transformation of the disc to be viewed as a Möbius left translation $z \mapsto z_0 \oplus z$ followed by a rotation. Here $\theta \in \Re$ is a real number, $z_0 \in D$, and $\bar{z}_0$ is the complex conjugate of z_0. The Möbius addition of two real numbers in the disc specializes to the Einstein velocity addition of parallel velocities in the special theory of relativity. A left Möbius translation is also called a left *gyrotranslation* [11]. Left gyrotranslations occur frequently in hyperbolic geometry [12], and are sometimes called hyperbolic pure translations.

The Möbius transformations of the disc D form a group, M. The rotations $z \mapsto e^{i\theta}z$ of the disc about its center form a subgroup R of M. In contrast, the left gyrotranslations $z \mapsto z_0 \oplus z$ of the disc do not form a subgroup of M. They do, however, form a twisted subgroup T of M. Furthermore, T is a transversal of R in M, $T^{-1} = T$ and T is normalized by R in M. Hence, the twisted subgroup T of M is a gyro-twisted subgroup. As such, according to Corollary 3.11, the groupoid $(T, \oplus)$ formed by the gyro-twisted subgroup T of R in M with its transversal operation $\oplus$ is a gyrogroup. This gyrogroup is studied in [10].

§4. Involutory Decompositions and Gyrocommutative Gyrogroups

Definition 4.1. (Involutory automorphisms) An automorphism of a group G is involutory if it equals its inverse automorphism.

The main result of this article is presented in the following theorem, demonstrating that gyrocommutative gyrogroups are associated with involutory automorphisms that groups in which they reside as subsets must possess.

Theorem 4.2. *Let $G = AH$ be a reduced gyro-decomposition of a group G, $H < G$. If A is a gyrocommutative gyrogroup, then there exists an involutory automorphism $\tau \in Aut(G)$ such that*

$$\tau(h) = h$$

for all $h \in H$, and

$$\tau(a) = a^{-1}$$

for all $a \in A$.

Proof. We define $\tau : G \to G$ by $\tau(g) = a^{-1}h$ for $g = ah \in G$, $a \in A$ and $h \in H$. Clearly, $\tau^2 = 1$, and τ is bijective. It remains to show that τ is a homomorphism. For this we need to use Theorem 5.5 of [11] according to which $h(a^{-1}, b^{-1}) = h(a, b)$ for $a, b \in A$ in any Gyrogroup A, and Theorem 5.9 of [11] according to which

$(a \odot b)^{-1} = a^{-1} \odot b^{-1}$ for $a, b \in A$ if and only if the gyrogroup A is Gyrocommutative. Let $g_1 = a_1h_1$ and $g_2 = a_2h_2$ be any two elements of G. On one hand

$$\begin{aligned}
\tau(g_1)\tau(g_2) &= \tau(a_1h_1)\tau(a_2h_2) \\
&= a_1^{-1}h_1a_2^{-1}h_2 \\
&= a_1^{-1}(a_2^{-1})^{h_1}h_1h_2 \\
&= a_1^{-1} \odot (a_2^{-1})^{h_1}h(a_1^{-1}, (a_2^{-1})^{h_1})h_1h_2 \\
&= a_1^{-1} \odot (a_2^{h_1})^{-1}h(a_1^{-1}, (a_2^{h_1})^{-1})h_1h_2 \\
&= a_1^{-1} \odot (a_2^{h_1})^{-1}h(a_1, a_2^{h_1})h_1h_2
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
\tau(g_1g_2) &= \tau(a_1h_1a_2h_2) \\
&= \tau(a_1a_2^{h_1}h_1h_2) \\
&= \tau(a_1 \odot a_2^{h_1}h(a_1, a_2^{h_1})h_1h_2) \\
&= (a_1 \odot a_2^{h_1})^{-1}h(a_1, a_2^{h_1})h_1h_2 \\
&= a_1^{-1} \odot (a_2^{h_1})^{-1}h(a_1, a_2^{h_1})h_1h_2
\end{aligned}$$

Hence,

$$\tau(g_1g_2) = \tau(g_1)\tau(g_2)$$

as desired. □

Theorem 4.2 suggests the following two definitions.

Definition 4.3. (Inverters and Stabilizers of Automorphisms in Groups) For any automorphism $\tau \in Aut(G)$ of a group G let the subset $K(\tau)$ and the subgroup $C(\tau)$ be given by

$$\begin{aligned}K(\tau) &= \{g \in G | \tau(g) = g^{-1}\}\\ C(\tau) &= \{g \in G | \tau(g) = g\}\end{aligned}$$

The subset $K(\tau)$ of G is called the *inverter* of τ in G, and the subgroup $C(\tau)$ of G is called the stabilizer of τ in G.

It follows from Theorem 4.2 that if $G = AH$, $H < G$, is a gyro-decomposition of a group G, and if A is a gyrocommutative gyrogroup, then there exists an automorphism $\tau \in Aut(G)$ such that $H \leq C(\tau)$, $A \subseteq K(\tau)$, and $G = K(\tau)C(\tau)$. This suggests the following definition and theorem.

Definition 4.4. (Involutory Decompositions) A gyro-decomposition (Def. 2.9) $G = AH$ of a group G is *involutory*, with respect to an involutory automorphism τ, if there exists an involutory automorphism $\tau \in Aut(G)$ such that $A \subseteq K(\tau)$ and $H \leq C(\tau)$.

Theorem 4.5. *Let $G = AH$ be a gyro-decomposition of a group G where (i) A is a twisted subgroup of G and (ii) H is a subgroup of G. Then, the decomposition $G = AH$ is involutory if and only if the transversal groupoid $(A, \odot)$ is a gyrocommutative gyrogroup.*

Proof. Let $G = AH$ be a gyro-decomposition of G where A is a twisted subgroup of G and H is a subgroup of G. To verify that involutory decomposition implies gyrocommutivity we assume that the decomposition is involutory. Since the decomposition is a gyro-decomposition, Def. 2.9, A is a gyrotransversal of H in G. Hence, by Def. 3.10, A is a gyro-twisted subgroup of G. Hence, by Corollary 3.11, $(A, \odot)$ is a gyrogroup whose operation is the transversal operation of A induced by G. It remains to verify that the gyrogroup $(A, \odot)$ is gyrocommutative. Let $a, b \in A$. On one hand

$$\begin{aligned}\tau(ab) &= \tau((a \odot b)h(a, b))\\ &= (a \odot b)^{-1}h(a, b)\end{aligned}$$

and on the other hand,

$$\begin{aligned}\tau(ab) &= \tau(a)\tau(b)\\ &= a^{-1}b^{-1}\\ &= (a^{-1} \odot b^{-1})h(a^{-1}, b^{-1})\end{aligned}$$

Hence, by the unique decomposition

$$(a \odot b)^{-1} = a^{-1} \odot b^{-1}$$

so that by Theorem 5.9 in [11], the gyrogroup A is gyrocommutative.

Conversely, let $G = AH$ be a gyro-decomposition of G as above. We now assume that the transversal groupoid $(A, \odot)$ is a gyrocommutative gyrogroup, and wish to show that the decomposition is involutory. This follows immediately from Theorem 4.2. $\square$

Thus, it follows from Theorems 4.2 and 4.5 that all gyrocommutative gyrogroups arise from the presence of involutory automorphisms. If a group G contains a gyrocommutative gyrogroup then there must exist an automorphism $\tau \in Aut(G)$ such that

$$K(\tau)C(\tau) = G \tag{4.1}$$

Hence, the existence of a gyrocommutative gyrogroup in a group is linked to the existence of an involutory automorphism of the group that decomposes it according to Eq. (4.1). The following Lemma provides us with an interesting special example that results from Theorem 4.5 when $A = K(\tau)$.

Theorem 4.6. *Let G be a group such that $G = K(\tau)H$ where $\tau \in Aut(G)$ and $H < G$. If*

(i) $H \leq C(\tau)$

(ii) $K(\tau)$ is a transversal of H in G

then the inverter $K(\tau)$ of τ in G with its transversal operation is a gyrocommutative gyrogroup.

Proof. By Theorem 4.5 we must show that $K(\tau)$ is a gyrotransversal of H in G, and a twisted subgroup of G. (i) $1_G \in K(\tau)$ since $\tau(1_G) = 1_G$. (ii) Clearly, if $a \in K(\tau)$ then $a^{-1} \in K(\tau)$. (iii) $K(\tau)$ is normalized by H for if $a \in K(\tau)$ and $h \in H$, then

$$\begin{aligned}\tau(hah^{-1}) &= \tau(h)\tau(a)\tau(h^{-1}) \\ &= ha^{-1}h^{-1} \\ &= (hah^{-1})^{-1}\end{aligned}$$

implying $hah^{-1} \in K(\tau)$ so that $H \subseteq N_G(K(\tau))$. It follows from $(i) - (iii)$ and the gyrotransversal definition that $K(\tau)$ is a gyrotransversal of H in G. Finally, if $a, b \in K(\tau)$, then $\tau(aba) = \tau(a)\tau(b)\tau(a) = a^{-1}b^{-1}a^{-1} = (aba)^{-1}$ so that $aba \in K(\tau)$. Hence $K(\tau)$ is a twisted subgroup of G, and the proof is complete. $\square$

We may note that the assumption $G = K(\tau)H$ in Theorem 4.6, where $K(\tau)$ is a transversal of H in G implies that τ is an involutory automorphism of G.

Definition 4.7. A nonempty subset X of a gyrogroup $(P, \odot)$ is a *subgroup* (of a gyrogroup), if it is a *group* under the restrriction of $\odot$ to X.

Definition 4.8. A subgroup X of a gyrogroup P is *normal* in P if
(i) $gyr[a, x] = 1$ for all $x \in X$ and $a \in P$.
(ii) $gyr[a, b](X) \subseteq X$ for all $a, b \in P$.
(iii) $a \bigodot X = X \bigodot a$ for all $a \in P$.

Lemma 4.9. *If X is a normal subgroup of a gyrogroup P, then P/X forms a factor gyrogroup.*

Proof. By (i)

$$(a \odot x) \odot y = a \odot (x \odot gyr[x,a]y) = a \odot (x \odot y)$$

for all $x, y \in X$, so the cosets $a \bigodot X$ partition P. Also by (iii), $x \odot b = b \odot y$ for some $y \in X$, so

$$\begin{aligned}(a \odot x) \odot b &= a \odot (x \odot gyr[x,a]b) = a \odot (x \odot b) \\ &= a \odot (b \odot y) = (a \odot b) \odot gyr[a,b]y) \in (a \odot b) \odot X\end{aligned}$$

as $gyr[a,b]y \in X$ by (ii), so the binary operation

$$(a \odot X) \odot (b \odot X) = (a \odot b) \odot X$$

is well defined, hence P/X forms a factor. $\square$

Definition 4.10. Let K be a twisted subgroup of a group G such that $G =< K >$, then the K-radical of G [1]

$$\Xi_K(G) = \{g \in G : g = k_1 \ldots k_n \text{ and } k_1^{-1} \ldots k_n^{-1} = 1 \text{ for some } k_i \in K, i = 1, 2 \ldots, n\}.$$

Theorem 4.11. *If $(P, \odot)$ is a gyrogroup, then P has a normal subgroup Ξ such that P/Ξ is a gyrocommutative gyrogroup (Note that Ξ is a group with group operation given by the resriction of $\odot$ to Ξ).*

Proof. Using the Representation Theorem 2.12, we can realize P as a gyrotransversal groupoid of a subgroup H of G. Replacing G by $\langle P \rangle$, we may assume $G = \langle P \rangle$, and replacing G by $G/C_H(P)$ we may assume the enveloping pair (G, H) is reduced, so that by Theorem 3.8, P is a twisted subgroup of G. Let $\Xi = \Xi_P(G)$ be defined as in Definition 4.10. By 2.1.2 in [1], $ax \in P$ for each $a \in P$ and $x \in \Xi$, so that $a \odot x = ax$ and (i) of Definition 4.8 holds for $X = \Xi$. Also the group operation of Ξ is the resriction of $\odot$ to Ξ, so that Ξ is a subgroup of P. By 2.1.1 of [1] Ξ is a normal subgroup of G. Hence (ii) of Definition 4.8 holds for $X = \Xi$. Also

$$(a \odot \Xi) = a\Xi = \Xi a = \Xi \odot a$$

so that (iii) of Definition 4.8 holds for $X = \Xi$. Hence Ξ is a normal subgroup of P. Let $\bar{G} = G/\Xi$, then $\bar{P}$ is a gyrotransversal of $\bar{H}$ in $\bar{G}$ and the corresponding gyrogroup is isomorphic to P/Ξ. By 2.2 in [1], there is a unique involutory automorphism τ of $\bar{G}$, such that $\bar{P} \subseteq K(\tau)$. Now for $\bar{h} \in \bar{H}$, as $\bar{h}$ acts on $\bar{P}$, $\tau^{\bar{h}}$ also inverts $\bar{P}$, so that by the uniqueness of τ, $\bar{h}$ centralizes τ. Hence by Theorem 4.5, P/Ξ is a gyrocommutive gyrogroup. $\square$

§5. Gyrocommutative Gyrogroups in Groups of Odd Order

Groups of odd order provide an abundant supply of gyrocommutative gyrogroups, as we see in the following theorem.

Theorem 5.1. *If G is a group of odd order possessing an involutory automorphism $\tau \in Aut(G)$. Then $G = PH$ where $H = C(\tau) \leq G$ is the stabilizer of τ in G, and where $P = K(\tau)$ is the inverter of τ in G. Furthermore, P is a transversal of H in G whose transversal groupoid $(P, \oplus)$ forms a gyrocommutative gyrogroup.*

If the inverter $P = K(\tau)$ of τ in G is not an Abelian normal subgroup of G, then $(P, \oplus)$ is not a subgroup of G.

Proof. Let $P = K(\tau)$ and $H = C(\tau)$ be the inverter and the stabilizer of τ in G, and let $f : G \to G$ be given by

$$f(x) = x^{-1}x^{\tau}$$

Then $f(x) = f(y)$ if and only if $y \in Hx$. Indeed,

$$\begin{aligned} f(x) = f(y) &\leftrightarrow x^{-1}x^{\tau} = y^{-1}y^{\tau} \\ &\leftrightarrow yx^{-1}x^{\tau} = y^{\tau} \\ &\leftrightarrow yx^{-1} = y^{\tau}(x^{\tau})^{-1} = (yx^{-1})^{\tau} \\ &\leftrightarrow yx^{-1} \in C(\tau) = H \\ &\leftrightarrow y \in Hx \end{aligned}$$

Let $G//H$ be the set of all right cosets of H in G, and let $\phi : G//H \to G$ be given by $\phi(Hx) = f(x)$ for all $x \in G$. It follows from the above mentioned property of f that ϕ is injective.

Moreover, $f(x) \in K(\tau)$ since

$$\begin{aligned} (f(x))^{\tau} &= (x^{-1}x^{\tau})^{\tau} \\ &= (x^{-1})^{\tau}x^{\tau^2} \\ &= (x^{\tau})^{-1}x \\ &= (x^{-1}x^{\tau})^{-1} \\ &= (f(x))^{-1} \end{aligned}$$

Hence, $\{f(x) : x \in G\} \subseteq P$.

For distinct $x, y \in P$, $Hx \neq Hy$, since if $x = hy$ then

$$y^{-1}h^{-1} = x^{-1} = \tau(x) = \tau(hy) = \tau(h)\tau(y) = hy^{-1}$$

so $h^y = h^{-1}$ and hence $y^2 \in C_G(h)$. Then as y is of odd order, $y \in C_G(h)$, so $h = h^y = h^{-1}$ and thus $h = 1$ as h is of odd order. Therfore,

$$|P| \geq |\phi(G/H)| = |G/H| \geq |P|$$

so that $\phi(G/H) = \{f(x) : x \in G\} = P$

Thus, the inverter $P = K(\tau)$ is a transversal of $H = C(\tau)$ in G. Hence, by Theorem 4.6, P is a gyrocommutative gyrogroup.

If P is a subgroup of G then, since it is a gyrotransversal of G, it is a normal subgroup (being normalized by H and by itself) and since it is gyrocommutative, it is Abelian. Hence, if P is not an Abelian normal subgroup of G then it is not a subgroup of G. $\square$

The inverter $P = K(\tau)$ and the stabilizer $H = C(\tau)$ of any involutory automorphism $\tau \in Aut(G)$, that give rise to the involutory decomposition $G = PH$ and to the gyrocommutative gyrogroup $(P, \oplus)$ in Theorem 5.1, are identified constructively in the following

Theorem 5.2. *Let G be a group of odd order possessing an involutory automorphism $\tau \in Aut(G)$, and let $g : G \to G$ be the map of G given by*

$$g(x) = x(x^\tau)^{-1}$$

$x \in G$. Then, the inverter $K(\tau)$ and the stabilizer $C(\tau)$ of τ in G are given by the equations

$$K(\tau) = \{g(x) : x \in G\}$$
$$C(\tau) = \{\sqrt{g(x)}x^\tau : x \in G\}$$

Proof. For all $x \in G$ we have the decomposition

$$x = \sqrt{g(x)}(\sqrt{g(x)}x^\tau) \tag{5.1}$$

Since
$$g(x)^\tau = x^\tau (x^{-1})^{\tau^2} = x^\tau x^{-1} = g(x)^{-1}$$
we have $g(x) \in K(\tau)$ for all $x \in G$. This, in turn, implies $\sqrt{g(x)} \in K(\tau)$ for all $x \in G$, so that

$$\{\sqrt{g(x)} : x \in G\} \subseteq K(\tau) \tag{5.2}$$

Similarly, since

$$(\sqrt{g(x)}x^\tau)^\tau = (\sqrt{g(x)})^\tau x^{\tau^2} = (\sqrt{g(x)})^{-1}x.$$

and since

$$(\sqrt{g(x)})^{-1}x = \sqrt{g(x)}x^\tau \tag{5.3}$$

as we will show below, we have

$$(\sqrt{g(x)}x^\tau)^\tau = \sqrt{g(x)}x^\tau$$

implying $\sqrt{g(x)}x^\tau \in C(\tau)$ for all $x \in G$, so that

$$\{\sqrt{g(x)}x^\tau : x \in G\} \subseteq C(\tau) \tag{5.4}$$

Eq. (5.3) follows from the chain of equations

$$\begin{aligned}\sqrt{g(x)}x^{\tau}x^{-1}\sqrt{g(x)} &= \sqrt{g(x)}(x(x^{\tau})^{-1})^{-1}\sqrt{g(x)} \\ &= \sqrt{g(x)}(g(x))^{-1}\sqrt{g(x)} \\ &= 1\end{aligned}$$

The equations in the Theorem for $K(\tau)$ and $C(\tau)$ follow from the uniqueness of the decomposition Theorem 5.1 and Eq. (5.1). □

Theorem 5.1 states that any group G of odd order with an involutory automorphism $\tau \in Aut(G)$ possesses the unique decomposition

$$G = K(\tau)C(\tau)$$

as the product (called the *gyrosemidirect* product) of

(i) the inverter $K(\tau)$ of τ in G, which is a gyrocommutative gyrogroup that sits inside G as a subset; and
(ii) the stabilizer $C(\tau)$ of τ in G, which is a subgroup of G.

Theorem 5.2 then identifies the inverter $K(\tau)$ and the stabilizer $C(\tau)$ of any involutory automorphism possessed by a group of odd order. The next natural step is to present in the following theorem the gyrogroup operation of the resulting gyrocommutative gyrogroup $K(\tau)$.

Definition 5.3. Let G be a group of odd order possessing an involutory automorphism $\tau \in Aut(G)$, and let $G = PH$ be the corresponding decomposition of G, where $P = K(\tau)$ and $H = C(\tau)$ are the inverter and the stabilizer of τ in G. Furthermore, let $\oplus$ be the transversal operation of the transversal P of H in G. Then

(i) the transversal groupoid (Def. 2.8) $(P, \oplus)$ is called the gyrocommutative gyrogroup generated by the pair (G, τ), and

(ii) the subgroup H of G is called the gyrations group generated by the pair (G, τ).

Theorem 5.4. *Let G be a group of odd order possessing an involutory automorphism $\tau \in Aut(G)$, let $(P, \oplus)$ be the gyrocommutative gyrogroup generated by (G, τ) and let H be the gyrations group generated by (G, τ). Then, the gyrogroup operation $\oplus$ of P is given by the equation*

$$x \oplus y = \sqrt{xy^2x}$$

and the gyrations of P are

$$gyr[x, y] = \alpha(\sqrt{xy^2x}\,x^{-1}y^{-1})$$

for all $x, y \in P$, where $\alpha(h)$ is the inner automorphism of P given by $\alpha(h)p = hph^{-1}$ for all $h \in H$ and $p \in P$.

Proof. Following Theorem 5.1 we have $G = PH$ where $P = K(\tau)$ and $H = C(\tau)$ are the inverter and the stabilizer of τ in G. Every $x \in G$ has the unique decomposition

$$x = \sqrt{g(x)}(\sqrt{g(x)}x^{\tau})$$

as a product of an element $\sqrt{g(x)} \in K(\tau)$ and an element $\sqrt{g(x)}x^{\tau} \in C(\tau)$.

Let $x, y \in P$ be two elements of P. Their product in G has the decomposition

$$\begin{aligned} xy &= (x \oplus y)h(x,y) \\ &= \sqrt{g(xy)}(\sqrt{g(xy)}(xy)^{\tau}) \end{aligned}$$

where, in P,

$$\begin{aligned} x \oplus y &= \sqrt{g(xy)} \\ &= \sqrt{xy((xy)^{\tau})^{-1}} \\ &= \sqrt{xy(x^{-1}y^{-1})^{-1}} \\ &= \sqrt{xyyx} \\ &= \sqrt{xy^2x} \end{aligned}$$

and in H,

$$\begin{aligned} h(x,y) &= \sqrt{g(xy)}(xy)^{\tau} \\ &= \sqrt{xy^2x}\,x^{-1}y^{-1} \end{aligned}$$

The elements $h(x,y) \in H$ give the gyrations $gyr[x,y]$ of the gyrogroup $(P, \odot)$ by $gyr[x,y] = \alpha(h(x,y))$ according to Def. 2.10, thus completing the proof. $\square$

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