Groups with all Cyclic Subgroups Conjugate - Permutable

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Dedicated to the memory of Michio Suzuki

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§1. INTRODUCTION

In this paper we continue the work on conjugate - permutable subgroups started in [1].

Definition 1.1. A subgroup H of a group G is a conjugate - permutable subgroups of G ($H \underset{C-P}{<} G$), if $HH^g = H^gH$ for all $g \in G$.

Definition 1.2. We denote by CCP the class of groups that every cyclic subgroup is conjugate - permutable.

In this paper we prove that a group G of prime exponent is in CCP if and only if it is a 2-Engel group. This leads to the proof that if G is a group, p an odd prime and $Syl_p(G) \subset CCP$, then

$$T_i = \{ x \in G | x^{p^i} = 1\}$$

is a normal subgroup of G. In the case of the even prime we show that if G is a group, $x \in G$, x a 2-element and $\langle x\rangle \underset{C-P}{<} G$, then $\langle x\rangle^G$ is a localy-finite 2-group. The above results lead to some new results and strengthening many results from [1]. For example, proving that any torsion group $G \in CCP$ is a locally-finite group, and if G has a finite odd exponent, then G is solvable.

§2. EXPONENT p-GROUPS AND PRELIMINARIES

For the reader's convenience and its importance for this section we include Lemma 1.6 of [1] below.

Lemma 2.1. *[1.6 of 1] If* $H \underset{C-P}{<} G$ *and* H *is a finite simple group, then for any* $g \in G$ $H = H^g$ *or* $[H, H^g] = \{1\}$.

Lemma 2.2. *If* G *is a group and* $|G| = p^3$ *for a prime* p, *then* $G \in CCP$.

Proof. Let $\langle x\rangle$ be a non-normal cyclic subgroup of G, then $|\langle x\rangle| = p$ and $\langle x\rangle$ is normal in a subgroup H of G where $|H| = p^2$. If $\langle x\rangle \neq \langle x\rangle^g$ for some $g \in G$, then $\langle x\rangle\langle x\rangle^g$ is a subset of H of order p^2 and $\langle x\rangle\langle x\rangle^g = H$. □

Lemma 2.3. *Let* G *be a group of exponent* p, *where* p *is a prime. Then* $G \in CCP$ *if and only if* $\langle x\rangle^G$ *is an abelian subgroup if and only if* G *is a 2-Engel group.*

Proof. By Lemma 2.1 and [2 or 12.3.6 of 7] □

Remark 2.4. If $G \in CCP$ and the exponent of G is p (p a prime), then for $p > 3$ the nilpotent class of $G \leq 2$ by [2 or 12.3.6 of 7] (and if G has n generators, then $|G| \leq p^{n+\binom{n}{2}}$). And for $p = 3$ the nilpotent class of $G \leq 3$ by [2 or 12.3.6 of 7] (and if G has n generators, then $|G| \leq 3^{n+\binom{n}{2}+\binom{n}{3}}$).

Remark 2.5. If G is a groups of exponent 3, then by [12.3.5 of 7] $G \in CCP$.

Definition 2.6. We denote by $T2$ the class of groups that every cyclic subgroup is 2-subnormal [5].

Lemma 2.7. *$G \in T2$ if and only if for all $x \in G$, $\langle x\rangle \lhd \langle x\rangle^G$, and therefore $T2 \subset CCP$.*

Proof. If for all $x \in G$, $\langle x\rangle \lhd \langle x\rangle^G$, then G is clearly in $T2$. Assume $G \in T2$, $x \in G$ and $\langle x\rangle$ is not normal in G. Then $\langle x\rangle \lhd H \lhd G$, but $\langle x\rangle^G \leq H$ so $\langle x\rangle \lhd \langle x\rangle^G$. Thus $\langle x\rangle$ and $\langle x\rangle^g$ are normal in $\langle x\rangle^G$ for all $g \in G$, so $\langle x\rangle\langle x\rangle^g = \langle x\rangle^g\langle x\rangle$. □

Definition 2.8. We denote by qH (quasi-Hamiltonian) the class of groups that every subgroup is permutable.

Remark 2.9. $qH \subset CCP$.

Remark 2.10. From [9, Exercise 2.3.8] Iwasawa's Theorem [9, Theorem 2.3.1], [9, Lemma 2.3.2] and Remark 2.9 we see that there is a 2-generators p-group $G \in CCP$ with nilpotent class of $G \geq n$ for any $n \in \mathbb{N}$.

Lemma 2.11. *If $G, H \in CCP$, then $H \times G \in CCP$, and therefore $T2 \cup qH \subsetneq CCP$.*

Proof. From [3] if $H \in T2$, then nilpotent class of $H \leq 4$ so by Remark 2.10 not all $G \in qH$ are in $T2$. Also in [4] one finds examples of $H \in T2$ that by [9, Theorem 2.4.14] are not in qH. □

§3. p-GROUPS

Note in this section the difference in the resulte between 2 and odd primes, in the 2 the results are not as strong.

Lemma 3.1. *If G is a group given $x, y \in G$ of order p prime, and $\langle x\rangle$ and $\langle y\rangle$ are conjugate - permutable subgroups of G, then $\langle x, y\rangle$ is a group of order $\leq p^3$ and of exponent p for p odd, and exponent ≤ 4 for $p = 2$.*

Proof. Assume that $[x, y] \neq 1$, then $[[x, y], x] = [[x, y], y] = 1$ (since y commutes with y^x and x commutes with x^y by Lemma 2.1. So, $[x, y]$ commutes with x and y. Thus, from above, and since $|\langle x, y\rangle| \geq p^3$, $\langle x, y\rangle$ is isomrphic to

$$H = \langle x, y : x^p = y^p = 1, [x, y]^x = [x, y] = [x, y]^y\rangle$$

which has order p^3 and exponent p for $p \neq 2$ and exponent 4 for $p = 2$ by [7, Exercise 5.3.6]. □

Theorem 3.2. *If G is a group, p an odd prime and $Syl_p(G) \subset CCP$, then*

$$T_i = \{\ x \in G | x^{p^i} = 1\}$$

is a normal subgroup of G.

Proof. Let x and $y \in T_1$, and $H = \langle x, y\rangle$, by Lemma 3.1 H has exponent p. Thus, $xy \in T_1$ and T_1 is a normal subgroup of G . Let $i + 1$ be the first case where T_{i+1} is not a normal subgroup of G. Let $\widetilde{G} = G/T_i$, by [1, Lemma 1.4] $\widetilde{T} = \{x \in \widetilde{G} : x^p = 1\}$ is a normal subgroup of $\widetilde{G}$, a contradiction. □

Proposition 3.3. *If G is a p-group of finite exponent where p is an odd prime and $G \in CCP$, then G is a solvable locally-nilpotent and locally-finite group.*

Proof. Assume that exponent $G = p^n$, and let T_i be as in Theorem 3.2, then

$$T_0 = 1 \lhd T_1 \lhd \ldots \lhd T_n = G,$$

now T_{i+1}/T_i is nilpotent of class ≤ 3 and is locally-finite by Remark 2.4, therefore G is solvable [7, 5.1.1] and locally-finite [7, 14.3.1], and since it is a locally-finite p-group it is a locally-nilpotent group. □

Proposition 3.4. *If G is a group, $x \in G$, x a 2-element and $\langle x \rangle \underset{C-P}{<} G$, then $\langle x \rangle^G$ is a localy-finite 2-group.*

Proof. Let $W = \{x_1, \ldots, x_n\}$ be a finite subset of $\langle x \rangle^G$. For all $1 \leq i \leq n$, $x_i \in \langle x \rangle^G$. So, $x_i = (x^{y_{1_i}})^{t_{1_i}} \ldots (x^{y_{m_i}})^{t_{m_i}}$, where $y_{j_i} \in G$ and $t_{j_i} \in \mathbb{Z}$, thus for $1 \leq i \leq n$,

$$x_i \in \langle x \rangle^{y_{1_i}} \ldots \langle x \rangle^{y_{m_i}} = K_i \ .$$

By [1, Lemma 1.1] for $1 \leq j$ and $i \leq n$ K_i and K_j are 2-subgroup of G that are permutable with each other. Therefore, $\langle W \rangle \leq K_1 \ldots K_n$ is a finite 2-subgroup of G. □

Proposition 3.5. *If G is a group and for some prime p, $Syl_p(G) \subset CCP$, then given $x, y \in G$ p-elements, xy is a p-element, and if p is odd then $o(xy) \leq max\{o(x), o(y)\}$.*

Proof. Suppose p is odd, and let $p^n = \max\{o(x), o(y)\}$, then x and $y \in T_n$ (where T_n is as in Theorem 3.2). By Theorem 3.2 T_n is a subgroup of G, so $xy \in T_n$. Thus, $o(xy) \leq p^n = \max\{o(x), o(y)\}$.

For $p = 2$, by Proposition 3.4 $\langle x \rangle^G$and$\langle y \rangle^G$ are localy-finite groups so by [7, 14.3.1] $K = \langle x \rangle^G \langle y \rangle^G$ is a localy-finite group. Thus, $H = \langle x, y \rangle$ is a finite group and $\langle x \rangle^H$ and $\langle y \rangle^H$ are 2 -groups by [1, lemma 1.3]

$H = \langle x \rangle^H \langle y \rangle^H$ is a 2-group. So, xy is a 2-element. □

Corollary 3.6. *Let $G \in CCP$ be a p-group. Then G is locally-finite and if p is odd then G is locally-nilpotent .*

Proof. For p odd, let W be any finite subset of G, and $exp(W) = p^n$ by Proposition 3.5

$$exp(\langle W \rangle) = exp(W) = p^n.$$

So, by Proposition 3.3 $\langle W \rangle$ is a finite p-group. For $p = 2$, let $W = \{x_1, \ldots, x_n\}$ be a finite subset of G. Then $W \subset \langle x_1 \rangle^G \ldots \langle x_n \rangle^G$ which is a localy-finite group by [7, 14.3.1] and Proposition 3.4. Therefore, $\langle W \rangle$ is a finite group. □

§4. THEOREMS

Theorem 4.1. *Let $G \in CCP$, and let P be a p-Sylow subgroup of G. Then:*
(i) P is a unique p-Sylow subgroup of G, and therefore normal.
(ii) If G is a torsion group, then G is a direct product of its Sylow subgroups.
(iii) If x a p-element and y a q-elment of G where p and q are distinct primes, then $xy = yx$.

Proof. By Proposition 3.5 $P = \{$all p-elements in $G\}$ is a unique p-Sylow subgroup of G, and therefore normal. □

Corollary 4.2. *If G is a group of finite odd exponent $G \in CCP$, then G is a solvable locally-nilpotent and locally-finite group .*

Proof. By Proposition 3.3 and Theorems 4.1. $\square$

Definition 4.3. Given a group G, $T(G) = \{a \in G | o(a) < \infty\}$.

Lemma 4.4. *If a torsion group G is a direct product of its Sylow subgroups and each of its Sylow subgroups is locally-finite, then G is a locally-finite group.*

Proof. Let W be a finite subset of G with exponent of $W = {p_1}^{t_1} \dots {p_n}^{t_n}$ where p_i is a prime for all i, and let P_i be the Sylow p_i subgroup of G. Then $\langle W \rangle \subset P_1 \dots P_n$ a locally-finite group by [7, 14.3.1], therefore $\langle W \rangle$ is finite. $\square$

Corollary 4.5. *If $G \in CCP$ is a torsion group, then G is a locally-nilpotent and locally-finite group.*

Proof. G is locally-finite by Theorems 4.1, Corollary 3.6, and Lemma 4.4, and thus by [1, Theorem 2.3] G is locally-nilpotent. $\square$

Theorem 4.6. *If $G \in CCP$, then $T(G) \lhd G$ and $T(G)$ is locally-finite.*

Proof. x and $y \in T(G)$. Let $x = x_{p_1} ... x_{p_n}$, and $y = y_{p_1} ... y_{p_n}$ (where x_{p_i} and y_{p_i} are p_i-element) from Theorem 4.1 $xy = (x_{p_1} y_{p_1}) .... (x_{p_n} y_{p_n})$, from Propsition 3.5, $o(xy) = lcm\{o(x_{p_1} y_{p_1}), ..., o(x_{p_n} y_{p_n})\} < \infty$. Thus, $T(G)$ is a torsion group in CCP and so by Corollary 4.5 $T(G)$ is locally-finite. $\square$

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