JOURNAL OF ALGEBRA **191,** 235–239 (1997)
ARTICLE NO. JA966924

Conjugate-Permutable Subgroups

Tuval Foguel*

Department of Mathematics, University of the West Indies, Mona, Kingston 7, Jamaica

Communicated by Walter Feit

Received September 6, 1996

INTRODUCTION

In the proof that a quasinormal subgroup is subnormal [4], one only needs to show that it is permutable with all of its conjugates. This leads to a new concept concerning subgroups.

DEFINITION. A subgroup H of a group G is a conjugate-permutable subgroup of $G(H <_{\mathrm{C-P}} G)$, if $HH^g = H^gH$ for all $g \in G$.

In the first section we prove that conjugate-permutable subgroups are subnormal, and we prove some elementary properties of conjugate-permutable subgroups. We also give examples of subnormal subgroups that are not conjugate-permutable subgroups, and of conjugate-permutable subgroups that are not quasinormal. Some of the results in the second section are:

THEOREM. *If G is a finite group and any cyclic subgroup H of any $P \in Syl_p(G)$, $H <_{\mathrm{C-P}} G$, then G is nilpotent.*

THEOREM. *If G is a locally finite group and for some prime p all cyclic subgroups of order a power of p are conjugate-permutable subgroups of G, then $P \in Syl_p(G)$, P is normal in G.*

THEOREM. *If G is a group containing a finite subgroup P such that $P \in Syl_p(G)$ and $P <_{\mathrm{C-P}} G$, then P is normal in G.*

THEOREM. *If G is a locally finite p-group p an odd prime such that any cyclic subgroup $H <_{\mathrm{C-P}} G$, then $T_i = \{x \in G \mid O(x) \le p^i\}$ is a normal subgroup of G.*

*E-mail: foguel@uwimona.edu.jm.

0021-8693/97 $25.00

THEOREM. *If G is a finite group and there exist $H <_{\text{C-P}} G$ such that H is a maximum subgroup of a $P \in Syl_p(G)$, then H or P is normal in G.*

1. PRELIMINARIES

Remark 1.1. Any quasinormal subgroup is a conjugate-permutable subgroup.

EXAMPLE 1.1. Let $H = \langle(12)(34)\rangle < S_4$. H is a conjugate-permutable subgroup of S_4, but H is not a quasinormal subgroup of S_4.

LEMMA 1.1. *If $H <_{\text{C-P}} G$ and $x_1, \ldots, x_n \in G$, then $H^{x_1}H^{x_2} \cdots H^{x_n}$ is permutable with any finite product of conjugates of H.*

LEMMA 1.2. *If $H <_{\text{C-P}} G$ and $H \leq K \leq G$, then $H <_{\text{C-P}} K$.*

LEMMA 1.3. *If $H <_{\text{C-P}} G$, G is a finite group, and $\{g_1, \ldots, g_n\}$ are a transversal of $N_G(H)$, then $H^G = H^{g_1}H^{g_2} \cdots H^{g_n}$ (note that if H is a p-group so is H^G).*

LEMMA 1.4. *If $H <_{\text{C-P}} G$ and f is a homomorphism of G, then $f(H) <_{\text{C-P}} f(G)$.*

LEMMA 1.5. *If H is a maximal conjugate-permutable subgroup of G, then $H \lhd G$.*

Proof. Suppose that H is not normal in G then $\exists\, g \in G$ such that $H^g \neq H$. Since $HH^g <_{\text{C-P}} G$, $HH^g = G$. Let $K = H^g$, now $g = hk$ where $h \in H$ and $k \in K$. Thus $K = H^g = H^{hk} = H^k$, so $K = K^{k^{-1}} = H$, a contradiction. ∎

COROLLARY 1.1. *If $H <_{\text{C-P}} G$ and G is a finite group, then H is subnormal in G.*

EXAMPLE 1.2. Let $D_8 = \langle x, y \mid x^8 = y^2 = 1,\ yxy = x^7\rangle$, $H = \langle y\rangle$, and $K = \langle yx^6\rangle$. Then H is subnormal in D_8 (since D_8 is nilpotent), but

$$HK = \{1, yx^6, y, x^6\} \neq \{1, yx^6, y, x^2\} = KH.$$

So H is not a conjugate-permutable subgroup.

COROLLARY 1.2. *If G is a finite group with all maximal subgroups conjugate-permutable subgroups, then G is nilpotent.*

LEMMA 1.6. *If $H <_{\text{C-P}} G$ and H is a finite simple group, then, for any $g \in G$, $H = H^g$ or $[H, H^g] = \{1\}$.*

Proof. Since $H <_{\text{C-P}} G$, $H <_{\text{C-P}} HH^g$, and HH^g is a finite group. Thus, by Corollary 1.1, H is subnormal in HH^g.

Let $K = H \cap H^g$ from above K is subnormal in H, so $K = H$ or $\{1\}$ (note that $K = H$ if and only if $g \in N_G(H)$). If H is a nonabelian simple group and $g \notin N_G(H)$, then, by [4, 13.3.1], $[H, H^g] = \{1\}$. If H is an abelian simple group and $g \notin N_G(H)$, then HH^g is an abelian group of order p^2, so $[H, H^g] = \{1\}$. ∎

COROLLARY 1.3. *If G is a finite group, $H <_{C-P} G$, and H is a simple group, then*

$$H^G \cong \underbrace{H \times H \times \cdots \times H}_{n\text{-times}},$$

where $[G: N_G(H)] = n$ (note that if H is nonabelian, then H^G is a minimal normal subgroup of G).

LEMMA 1.7. *If H is a nonabelian simple subnormal subgroup of a finite group G, then $H <_{C-P} G$.*

Proof. By [4, 13.3.1], for any $y \in G$, $H = H^g$ or $[H, H^g] = \{1\}$. So $HH^g = H^gH$ and $H <_{C-P} G$. ∎

Remark 1.2. If H is an abelian simple subnormal subgroup of a finite group G, then H need not be a conjugate-permutable subgroup of G, for example, H in Example 1.2.

2. THEOREMS

THEOREM 2.1. *If G is a group containing a finite subgroup P such that $P \in Syl_p(G)$ and $P <_{C-P} G$, then P is normal in G.*

Proof. For all $g \in G$, $|PP^g|$ is a p-number. Thus $PP^g \in Syl_p(G)$ and $P \lhd G$. ∎

THEOREM 2.2. *If G is a finite group and there exist $H <_{C-P} G$ such that H is a maximum subgroup of a $P \in Syl_p(G)$, then H or P is normal in G.*

Proof. Assume that H is not normal in G. By Lemma 1.3 H^G is a p-group and since $H \neq H^G$, $H^G \in Syl_p(G)$. ∎

COROLLARY 2.1. *If G is a finite group and there exist $H <_{C-P} G$ such that H is a maximum subgroup of a $P \in Syl_2(G)$, then G is solvable.*

Proof. H^G is a 2-group, and $|G/H^G| = 2m$ or m where m is odd. Thus H^G and G/H^G are solvable, so G is solvable. ∎

LEMMA 2.1. *If G is a finite group and for some prime p all cyclic subgroups of order a power of p are conjugate-permutable subgroups of G, then $P \in Syl_p(G)$, P is normal in G.*

Proof. Let $P \in Syl_p(G)$, $P = \{x_1, \dots, x_n\}$. By Lemma 1.3 for all i, $\langle x_i \rangle^G$ is a normal p-subgroup of G. Therefore $H = \langle x_1 \rangle^G \cdots \langle x_n \rangle^G$ is a normal p-subgroup of G and $P \leq H$, so P is normal in G. ∎

THEOREM 2.3. *If G is a finite group and any cyclic subgroup H of any $P \in Syl_p(G)$, $H <_{\mathrm{C-P}} G$, then G is nilpotent.*

Proof. By Lemma 2.1 for any prime $P \in Syl_p(G)$, P is normal in G. ∎

THEOREM 2.4. *If G is a locally finite group and for some prime p all cyclic subgroups of order a power of p are conjugate-permutable subgroups of G, then $P \in Syl_p(G)$, P is normal in G.*

Proof. Let x and y be p-elements of G, and $H = \langle x, y \rangle$, H is a finite group satisfying Lemma 2.1 so xy is a p-element. Therefore $S = \{x \in G: x$ is a p-element$\}$ is a normal subgroup of G. ∎

LEMMA 2.2. *If G is a finite p-group p an odd prime with all subgroups of order p are conjugate-permutable subgroups of G, then $T = \{x \in G: O(x) \leq p\}$ is a normal subgroup of G.*

Proof. By Corollary 1.3 for all $x \in T$, $\langle x \rangle^G$ is a normal elementary abelian subgroup of G. Let $x, y \in T$, by [4, 5.2.8], $\langle x \rangle^G \langle y \rangle^G$ is a nilpotent group of class at most 2. Therefore, by [4, 5.3.5],

$$(xy)^p = x^p y^p [x, y]^{\binom{p}{2}},$$

but since $[x, y] \in \langle x \rangle^G \cap \langle y \rangle^G$, $O([x, y]) = p$ or 1. Thus $(xy)^p = x^p y^p = 1$, and T is a normal subgroup. ∎

LEMMA 2.3. *If G is a locally finite p-group p an odd prime with all subgroups of order p are conjugate-permutable subgroups of G, then $T = \{x \in G: O(x) \leq p\}$ is a normal subgroup of G.* ∎

Proof. Let x and $y \in T$, and $H = \langle x, y \rangle$, H is a finite group satisfying Lemma 2.2, so $(xy)^p = 1$. ∎

LEMMA 2.4. *If G is a locally finite p-group p an odd prime with all subgroups of order p are conjugate-permutable subgroups of G, then given $x, y \in G$ of order p $\langle x, y \rangle$ is a group of order $\leq p^3$.*

Proof. Let $H = \langle x, y \rangle$. Assume that $[x, y] \neq 1$, then $[x, y] \in \langle x \rangle^H \cap \langle y \rangle^H$. So $[x, y]$ commutes with x and y. Thus

$$H = \langle x, y: x^p = y^p = 1, [x, y]^x = [x, y] = [x, y]^y \rangle$$

has order p^3. ∎

THEOREM 2.5. *If G is a locally finite p-group p an odd prime such that any cyclic subgroup $H <_{C-P} G$, then $T_i = \{x \in G \mid O(x) \leq p^i\}$ is a normal subgroup of G.*

Proof. By Lemma 2.3, T_1 is a normal subgroup of G. Let $i + 1$ be the first case where T_{i+1} is not a normal subgroup of G. Let $\tilde{G} = G/T_i$, by Lemma 2.3, $\tilde{T} = \{x \in \tilde{G}: O(x) \leq p\}$ is a normal subgroup of $\tilde{G}$, a contradiction. ∎

COROLLARY 2.2. *If G is a locally finite group and for some odd prime p all cyclic subgroups of order a power of p are conjugate-permutable subgroups of G, then $T_i = \{x \in G \mid O(x) \leq p^i\}$ is a normal subgroup of G.*

Proof. By Theorems 2.4 and 2.5 and the fact that T_i is a normal subset of G. ∎

REFERENCES

1. T. Foguel, On seminormal subgroups, *J. Algebra* **165**(3) (1994), 633–636.
2. O. H. Kegel, Produkte nilpotenter gruppen, *Arch. Math.* **12** (1961), 90–93.
3. O. Ore, On the application of structure theory to groups, *Bull. Amer. Math. Soc.* **44** (1938), 801–806.
4. D. J. S. Robinson, ‘‘A Course in the Theory of Groups,’’ Springer-Verlag, New York, 1982.
5. J. J. Rotman, ‘‘The Theory of Groups, An Introduction,’’ 3rd ed., Allyn and Bacon, Boston, 1984.
6. X. Su, Seminormal subgroups of finite groups, *J. Math* (*Wuhan*) **8**(1) (1988), 5–10.
7. P. Wang, Some sufficient conditions of a nilpotent groups, *J. Algebra* **148** (1992), 289–295.