Problem 1: Show that every planar graph has a vertex of degree at most 5.

Proof. We will prove this statement by using a proof by contradiction. We will assume that G is planar and that all vertices of G have degree greater than or equal to 6. We know from Theorem 1 of 1.3 that the sum of the degrees of all verticies of a graph is equal to twice the number of edges. Since each vertex has a degree of 6 or greater, we obtain

$$\begin{aligned} 6v &\leq 2e \\ 3v &\leq e \end{aligned} \tag{1}$$

where v is the number of vertices in the graph and e is the number of edges. If G is planar, then it must satisfy the Corollary of Euler's Formula. Thus, the following must be true:

$$e \leq 3v - 6 \tag{2}$$

From (1) and (2) we get

$$3v \leq e \leq 3v - 6 \tag{3}$$

The inequality (3) is impossible since $3v > 3v - 6$. Therefore, we have reached a contradiction to our assumption that all vertices of G have degree greater than or equal to 6. Hence, we conclude if a graph G is planar, then G has a vertex of degree at most 5. □

Problem 2: Show that an n-vertex graph cannot be bipartite graph if it has more than $\frac{1}{4}n^2$ edges.

Proof. We will prove that if a graph G is bipartite then it cannot have more than $\frac{1}{4}n^2$ edges. We will prove this by contradiction; therefore, we will assume that G is bipartite and that G has more than $\frac{1}{4}n^2$ edges.
Let m be the number of verticies on one side of the bipartite graph G; thus $n - m$ is the number of verticies on the other side. The number of edges in the complete $K_{m,n-m}$ bipartite graph is $m(n - m)$. Hence, the number of edges in G is less than or equal to $m(n - m)$, i.e.

$$e \leq m(n - m) = mn - m^2 \tag{4}$$

where n is a fixed integer and e is the number of edges in the graph G. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x(n - x)$. The restriction of this function over the positive integers will give as the quantity $m(n - m)$. We can find the maximum of the function $f(x) = xn - x^2$, $x \in \mathbb{R}$, using the first and second derivatives.

$$\begin{aligned} f'(x) &= n - 2x \\ f''(x) &= -2 \end{aligned}$$

By setting $f'(x) = 0$, we obtain $x = \frac{n}{2}$, and since $f''(x) < 0$ we know that f has a maximum at $x = \frac{n}{2}$. We can now calculate the maximum value of $f(x)$.

$$f\left(\frac{n}{2}\right) = \frac{n}{2}\cdot n - \left(\frac{n}{2}\right)^2 = \frac{1}{4}n^2$$

By using equation(4) we obtain

$$e \leq mn - m^2 \leq \frac{1}{4}n^2 \tag{5}$$

Our assumption that G has more than $\frac{1}{4}n^2$ edges is in contradiction with (5). Hence, we have now proven that an n-vertex graph cannot be bipartite if it has more than $\frac{1}{4}n^2$ edges. □

Problem 3: Show that a planar graph G with 8 verticies and 13 edges cannot be 2-colored.

Proof. Let G be a planar graph G with $v = 8$ vertices and $e = 13$ edges and G can be 2-colored. Since G can be 2-colered, then G does not have a circuit of length 3. Using Problem 6 from Test 1, G must satisfy the following inequality $e \leq 2v - 4$. But

$$\begin{aligned} e &\leq 2v - 4 \qquad (6)\\ 13 &\leq 2(8) - 4 \\ 13 &\leq 12 \end{aligned}$$

We have reached a contradiction. Thus G cannot be 2-colored. □

Problem 4: Suppose a tree T has an even number of edges. Show that at least one vertex of T must have even degree.

Proof. We will prove this statement by using a proof by contradiction. We will assume that a tree T has an even number of edges and that all verticies of T have odd degree. Let e be the number of edges of T and v be the number of vertices of T. By Theorem 2 in Section 3.1, $e = v - 1$. Since e is an even number, then v must be odd. Hence T has odd number of vertices of odd degree which contradicts the Corollary on page 23. (In any graph, the number of vertices of odd degree is even.) Therefore T has at least one vertex of even degree. □

Problem 5: Show that a graph is connected if and only if it has a spanning tree.

Proof. We will prove that if a graph G is connected, then it has a spanning tree. We will assume that a graph is connected. We know that for a graph to be connected, there must exist a path from any vertex a to any vertex b. We will remove an edge from G until we have no circuits without disconnecting the graph. The end product will therefore be a spanning tree. We have now proven that if G is connected then it has a spanning tree.

We will now prove the opposite direction of the biconditional statement. We will prove that if a graph G has a spanning tree, then G is a connected graph. Let T be a spanning tree for G. Then T is connected and contains all vertices of G. Therefore G is connected. □

Problem 6: How many positive integers less than 1,000,000 are there with distinct digits and divisible by 5?

Solution: We will determine how many positive integers there are with distinct digits that are less than 1,000,000 and divisible by 5. Let us consider six cases where the number has 1, 2, 3, 4, 5, or 6 digits.

- Case 1: The number has 1 digit. There is only one possible integer with 1 digit that is divisible by 5, which is 5.
- Case 2: 2 digits. If the number ends with a 0 there are 9 possible digits for the first digit (1-9). If the number ends with a 5 there are 8 possibilities for the first number (1-9 except 5). Thus, we obtain that there are $(9*1)+(8*1)=17$ integers with 2 distinct digits that are divisible by 5.
- Case 3: 3 digits. If the number ends with a 0 there are 9 possible digits for the first digit and 8 possibilities for the second. If it ends with a 5 there are 8 possibilities for the first and also 8 for the second. Therefore, there are $(9*8*1)+(8*8*1)=136$ distinct numbers.
- Case 4: 4 digits. Following the same pattern, we'll obtain $9*8*7*1$ possible integers if the number ends with a 0 and $8*8*7*1$ if it ends with a 5. Adding the products gives us 952.
- Case 5: 5 digits. If the integer ends in a 0, we have $9*8*7*6*1$ possible values and if it ends with a 5 we have $8*8*7*6*1$ possible values. We obtain 5712 by adding the two products.
- Case 6: 6 digits. By continuing the same logic, we obtain $(9*8*7*6*5*1)+(8*8*7*6*5*1)=$ 28560 distinct six digit numbers that are divisible by 5.

We can add up the six disjoint cases to obtain a total of $1+17+136+952+5712+28560=35378$ integers divisible by 5 that contain distinct digits and are divisible by 5.

Problem 7: A secretary works in a building located m blocks east and n blocks north of his home. Every day he walks $m+n$ blocks to work. How many different routes are possible for him? (All streets are either parallel or perpendicular to each other.)

Solution: Since we know that the secretary walks $m+n$ blocks each day, we know that he must take m blocks going east, and n blocks going north. He will never go west or south. Thus, we can categorize each "block" that the man walks in two ways: north (N) and east (E). Then each route corresponds to a sequence of m Es and n Ns. Thus the total number of routes is $P(m+n,m,n)$, permutations with repetitions, since E is repeated m times and N is repeated n times. This number is the same as $C(m+n,m)=C(m+n,n)$ because

$$\mathrm{C}(m+n,m)=\frac{(m+n)!}{((m+n)-m)!\cdot m!}=C(m+n,n)=P(m+n,m,n)$$

Problem 8: How many solutions are there to the equation

$$x_1+x_2+x_3+x_4=29$$

where x_1, x_2, x_3, and x_4 are integers such that $x_1\geq 0, x_2>3, x_3\geq 2$, and $1\leq x_4<4$.

Solution: We can rewrite the ranges for the variables in the following form:

$$\begin{array}{rcccl} 0 & \leq & x_1 & & \\ 4 & \leq & x_2 & & \\ 2 & \leq & x_3 & & \\ 1 & \leq & x_4 & \leq & 3 \end{array}$$

Let $y_1 = x_1$, $y_2 = x_2 - 4$, $y_3 = x_3 - 2$, and $y_4 = x_4 - 1$. With this, we will obtain the following:

$$\begin{aligned} y_1 + (y_2 + 4) + (y_3 + 2) + (y_4 + 1) &= 29 \\ y_1 + y_2 + y_3 + y_4 &= 22 \end{aligned}$$

Where $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$, and $0 \leq y_4 \leq 2$. Let us consider 3 cases where $y_4 = 0$, $y_4 = 1$, and $y_4 = 2$.

- **Case 1** ($y_4 = 0$): The equation is now $y_1 + y_2 + y_3 = 22$. None of the variables can exceed 22 nor fall under 0. We can use combinations to solve this problem, where we must choose 22 objects from 3 types of objects. $C(22 + 3 - 1, 22)$ gives us the number of combinations for the sum of y_1, y_2, and y_3 to equal 22.
- **Case 2** ($y_4 = 1$): The equation is $y_1 + y_2 + y_3 = 21$. Using the same logic from case (1), we obtain $C(21 + 3 - 1, 21)$ possible solutions for y_1, y_2, and y_3.
- **Case 3** ($y_4 = 2$): The equation is $y_1 + y_2 + y_3 = 20$. There are obtain $C(20 + 3 - 1, 20)$ possible solutions for the sum of y_1, y_2, and y_3 to return 20.

Adding the three disjoint cases will give us the total number of solutions for $y_1 + y_2 + y_3 + y_4 = 22$ such that each $y_1, y_2, y_3, y_4 \geq 0$. The same number of solutions will apply when we substitute to obtain $x_1 + x_2 + x_3 + x_4 = 29$ such that $x_1 \geq 0$, $x_2 > 3$, $x_3 \geq 2$, and $1 \leq x_4 < 4$.

$$C(24, 22) + C(23, 21) + C(22, 20) = \frac{24!}{2! \cdot 22!} + \frac{23!}{2! \cdot 21!} + \frac{22!}{2! \cdot 20!}$$

Problem 9: How many one-to-one functions are there from a set with m elements to a set with n elements? (A function $f : A \to B$ is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all $a, b \in A$.)

Solution: Let $A = \{a_1, a_2, \ldots, a_m\}$ be the domain and $B = \{b_1, b_2, \ldots, b_n\}$ be the codomain of f. We will consider the cases where $m \leq n$ and $m > n$.

- **Case 1** ($m \leq n$): Since any two distinct elements in A must have distinct images in B, and $|B| = n$, we know that there are n possible images for $a_1 \in A$, $n - 1$ possible images for $a_2 \in A$, and so on, and $n - m + 1$ possible images for $a_m \in A$. Therefore, the total number of one-to-one functions from a set A where $|A| = m$ to a set B where $|B| = n$ is the following:

$$(n) \cdot (n-1) \cdot \ldots \cdot (n - m + 1) = \frac{n!}{(n-m)!}$$

- **Case 2** ($m > n$): Like Case 1, any two distinct elements in A must be mapped to distinct elements in B. Since $|A| > |B|$, a_1 will have n possible images, a_2 will have $n-1$ possibilities, down to a_n which will have 1 possible image because the elements $a_1, a_2, \ldots, a_{n-1}$ already are mapped to $n-1$ elements in B, leaving 1 remaining out of the n elements in B. Because $m > n$, there exists $a_{n+1} \in A$; however, there are no remaining elements in B, so there are 0 possible images for a_{n+1}. Therefore, there are 0 one-to-one functions when $m > n$.

Problem 10: How many onto functions are there from a set with 8 elements to a set with 4 elements? (A function $f : A \to B$ is called *onto*, or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.)

Solution: We will let the domain be the set A, and $|A| = 8$. Let the codomain be the set $B = \{b_1, b_2, b_3, b_4\}$. We will define sets of functions such that $B_i = \{f : A \to B \mid (\forall x \in A) f(x) \neq b_i\}$.

$$\begin{array}{rcl} B_1 & = & \{f : A \to B \mid (\forall x \in A)\ f(x) \neq b_1\} \\ B_2 & = & \{f : A \to B \mid (\forall x \in A)\ f(x) \neq b_2\} \\ B_3 & = & \{f : A \to B \mid (\forall x \in A)\ f(x) \neq b_3\} \\ B_4 & = & \{f : A \to B \mid (\forall x \in A)\ f(x) \neq b_4\} \end{array}$$

The union $B_1 \cup B_2 \cup B_3 \cup B_4$ represents a set of functions where

$$\left\{ \begin{array}{lll} f : A \to B \mid & ((\forall x \in A)\ f(x) \neq b_1) & \text{OR} \\ & ((\forall x \in A)\ f(x) \neq b_2) & \text{OR} \\ & ((\forall x \in A)\ f(x) \neq b_3) & \text{OR} \\ & ((\forall x \in A)\ f(x) \neq b_4) & \end{array} \right\}.$$

We want to find the size of this union because its complement ($\overline{B_1 \cup B_2 \cup B_3 \cup B_4}$) is the set

$$\left\{ \begin{array}{lll} f : A \to B \mid & ((\exists x \in A)\ f(x) = b_1) & \text{AND} \\ & ((\exists x \in A)\ f(x) = b_2) & \text{AND} \\ & ((\exists x \in A)\ f(x) = b_3) & \text{AND} \\ & ((\exists x \in A)\ f(x) = b_4) & \end{array} \right\}$$

which is, consequently, the number of onto functions of $f : A \to B$. We first need to find $|B_1 \cup B_2 \cup B_3 \cup B_4|$ to find the number of elements in its complement.

$$\begin{array}{rcl} |B_1 \cup B_2 \cup B_3 \cup B_4| & = & (|B_1| + |B_2| + |B_3| + |B_4|) - (|B_1 \cap B_2| + |B_1 \cap B_3| + |B_1 \cap B_4| + |B_2 \cap B_3| + |B_2 \cap B_4| + |B_3 \cap B_4|) + (|B_1 \cap B_2 \cap B_3| + |B_1 \cap B_2 \cap B_4| + |B_1 \cap B_3 \cap B_4| + |B_2 \cap B_3 \cap B_4|) - |B_1 \cap B_2 \cap B_3 \cap B_4| \\ \\ & = & (3^8 \cdot 4) - (2^8 \cdot 6) + (1^8 \cdot 9) - 0 \\ & = & 24711 \end{array}$$

Finally, 4^8 is the total number of functions of $f : A \to B$ since each of the 4 elements in A can map to any of 8 elements in B. Therefore, we can find the number of onto functions in the graph with the following equation:

$$\begin{array}{rcl} |\overline{B_1 \cup B_2 \cup B_3 \cup B_4}| & = & 4^8 - |B_1 \cup B_2 \cup B_3 \cup B_4| \\ & = & 4^8 - 24712 \\ & = & 40824 \end{array}$$