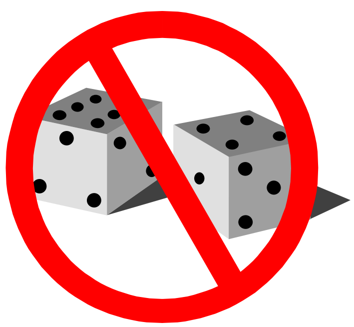
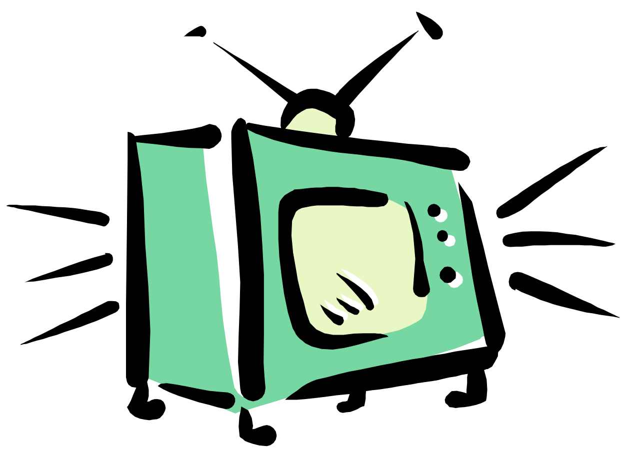
Probability & Statistics Through Game Shows





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Websites

`http://www.amstat.org/publications/jse/v9n3/biesterfeld.html`

Dr. Amy Biesterfeld, a former graduate student colleague of Matt's, wrote an article on *The Price Is Right* for the *Journal of Statistical Education*. Amy's article includes *Plinko*, *Master Key*, and *The Range Game*. Some suggested activities in the article are suitable for high school, but some are also quite advanced.

`http://gscentral.net/pricing.htm`

Brad Francini has compiled a complete directory of games on *The Price Is Right*. His website, part of Game Show Central.net, has game descriptions, countless photographs from the show, and odds of winning some games. We'd like to thank Brad for allowing us to use his descriptions.

`http://www.stat.sc.edu/~west/javahtml/LetsMakeaDeal.html`

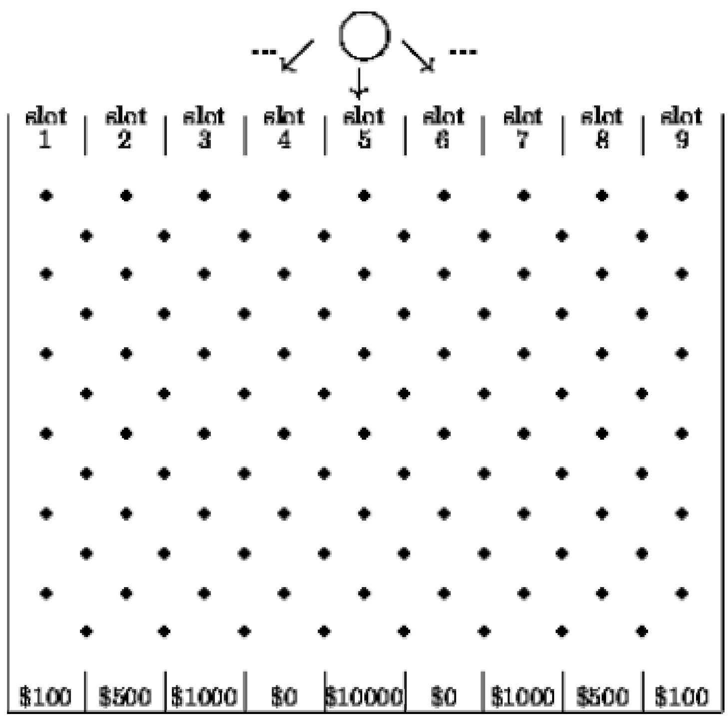
As the name suggests, students can simulate the famous Monty Hall problem with this Java script. You can also access this website from `http://stat201.isfun.net/` by Prof. R. Webster West, which boasts scores of applets and other simulation tools.

`http://www.cut-the-knot.org/hall.shtml`

Cut-the-Knot.org provides another simulation to the Monty Hall problem. Only explore this website when you have a few hours free! Cut-the-Knot.org has hundreds of web-based resources for mathematics teachers, including a massive section on probability.

Plinko (*The Price Is Right*)

Game description: A contestant is shown four small prizes. For each prize, s/he sees two digits, but only one is right. Either the first digit shown is the correct first digit or the last digit shown is the correct last digit. For each small item guessed correctly, the contestant wins a chip. Bob gives the contestant a free chip just for playing the game, so five chips can be earned. The contestant climbs to the top of the Plinko board (see graphic below) and drops the chips one at a time. The pegs send the chip bouncing all over the board until they land in slots representing money amounts at the bottom. The slots are, from left to right; $100, $500, $1000, $0, $10000, $0, $1000, $500, $100. After all the chips have dropped, the contestant wins whatever money s/he has earned to that point.

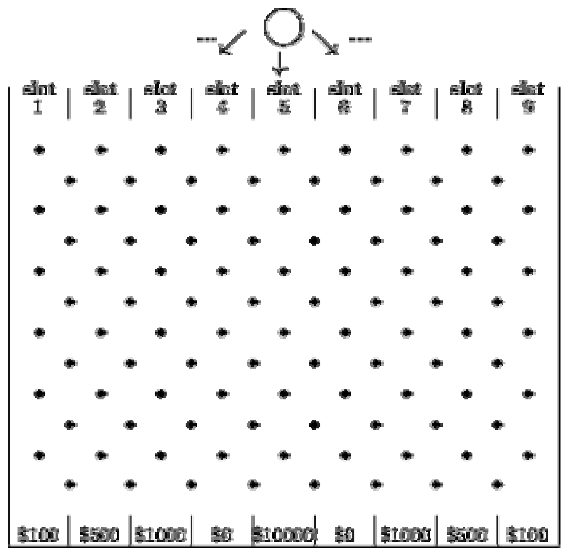


The Plinko board has 12 levels. If you start from Slot 5, for example, you need 6 left bounces and 6 right bounces to hit the $10,000 slot at the bottom.

Set-up: None. Students will need their TI-83 calculators.

Learning objectives: probability, random variables, expected value and standard deviation, the binomial distribution, linear combinations

Plinko



1. You can simulate one "drop" with the random integer function by using randInt(0,1,12) [STO] $\rightarrow$ L_1 on the TI-83. If each 0 counts as a left bounce and each 1 as a right bounce, the calculator creates 12 values randomly of either 1 or 0 and stores them in List 1.
2. Tally and record the total number of right bounces. This can be recorded as sum(L_1) from the "home screen."
3. To find your location at the bottom of the Plinko board, use the conversion table below (which accounts for reflections off the wall). The table assumes you drop the Plinko chip from Slot 5.

# of "right bounces"		winnings
6	$\rightarrow$	$10,000
0,4,8,12	$\rightarrow$	$1,000
1,3,9,11	$\rightarrow$	$500
2,10	$\rightarrow$	$100
5,7	$\rightarrow$	$0

4. Repeat steps 1-3 five times. Record your results in the table below.

Plinko chip #	# of right bounces	amount you win
1		$
2		$
3		$
4		$
5		$
YOUR TOTAL WINNINGS:		$

- Record the results of your five chips, along with those of your classmates, in the table below.

amount won	$0	$100	$500	$1,000	$10,000
frequency					

- Change these frequencies into probabilities.

amount won	$0	$100	$500	$1,000	$10,000
probability					

- Compare your results to the "true values":

x	0	100	500	1,000	10,000
$p(x)$	396/1024	33/1024	116/1024	248/1024	231/1024

Why are the simulation results not identical to the "true values"?

- What happens if you drop the chip from Slot 4? Using the diagram of the Plinko board, see if you can deduce which bounces give which dollar amounts. (The first answer has been provided for you.)

# of right bounces		winnings
7	→	$10,000
	→	$1,000
	→	$500
	→	$100
	→	$0

Plinko, Part II

- Use your simulation data from before to calculate the expected winnings from one Plinko chip.

- Use the table of "true values" to calculate the same expected value.

x	0	100	500	1,000	10,000
$p(x)$	396/1024	33/1024	116/1024	248/1024	231/1024

- Do the same for standard deviation.

The table below indicates the probability distribution for Plinko if you drop a chip from Slot **4** rather than Slot 5.

x	0	100	500	1,000	10,000
$p(x)$	355/1024	58/1024	157/1024	256/1024	198/1024

- Calculate the expected value of this probability distribution.

- Calculate the standard deviation.

- Which slot wins you more money in the long run: Slot 4 or Slot 5?

- Which slot yields less variability: Slot 4 or Slot 5?

Plinko, Part III

Recall the simulation of bounces on a Plinko board. Consider the random variable y defined by

y = the number of 1's in twelve 0/1 simulations

- What are the possible values of y?

- What type of random variable is y: continuous or discrete?

- What is the <u>name</u> of the distribution of y? Give the values of n and p.

- Use the probability distribution of y, along with the conversion table for Slot 5 (see below), to construct the probability distribution for your winnings on one Plinko chip. Remember that the table below accounts for reflections off the side walls.

# of "right bounces" (y)		winnings (x)
6	$\rightarrow$	\$10,000
0,4,8,12	$\rightarrow$	\$1,000
1,3,9,11	$\rightarrow$	\$500
2,10	$\rightarrow$	\$100
5,7	$\rightarrow$	\$0

x	0	100	500	1,000	10,000
$p(x)$					

Compare your answers to the table provided earlier to see if you are correct.

Plinko, Part IV

In *Plinko*, contestants can earn up to a total of 5 Plinko chips. Let the amounts won from these five chips be X_1, X_2, X_3, X_4, and X_5. Suppose Bob Barker now offers us <u>two</u> ways to play *Plinko*: we can drop 5 chips as usual and take the combined winnings, or we can drop 1 chip and win five times the value of that chip. That is, your total winnings T can either be

i. $T = X_1 + X_2 + X_3 + X_4 + X_5$ (the usual way), or
ii. $T = 5X_1$ (Bob's special offer)

We know from past computation that $\mu_X = \$2,557.91$ and $\sigma_X = \$4,035.66$.

- What are the expected payoffs for each of these two options?

- What are the standard deviations for each of these two options?

- Which option would you take?

The Grand Game (*The Price Is Right*)

Game description: The contestant starts with a bank of $1. Six grocery items are shown along with a target price for the grocery items. Four items are below the target price, two are above. Each time the contestant picks an item under the target price, s/he earns a "0" in the bank (i.e., the contestant's bank goes from $10 to $100 to $1000 to a possible total of $10,000). If the contestant picks all four items under the target price, s/he wins $10,000 in cash. If the contestant picks one of the items above the target price, the game is over. Should the contestant lose the game with $1, $10, or $100 in the bank, s/he wins that amount. However, when the contestant gets to $1,000 (when there are one good item and two bad items left), Bob ups the stakes. The contestant can quit and keep the $1,000 or risk it and go on for the win. If the contestant picks the final item, obviously s/he wins the $10,000, but if s/he picks a bad item, s/he loses the game and s/he loses the $1,000 as well!

Set-up: None.

Learning objectives: random variables, expected value, probability versus expected value

The Grand Game

A contestant has won $1,000. Two of the remaining products cost more than the target price, and only one costs less than the target price.

- What is the probability the contestant will pick the one remaining product that costs less than the target price?

- Based on this probability alone, would you risk your $1,000?

- The contestant wins $10,000 if he picks the right product, and s/he drops to $0 if s/he picks either of the wrong products. Let x represent the contestant's winnings, assuming s/he risks $1,000. Fill in the table below.

x	
$p(x)$	

- Based on the table, calculate the contestant's expected winnings if s/he risks $1,000.

- Would you risk $1,000 in *The Grand Game* for a chance at $10,000?

Let 'Em Roll (*The Price Is Right*)

Game description: The contestant earns a roll of five dice to start the game and can earn two more rolls by pricing three grocery items. The price of a first grocery item is shown and the contestant must guess if the next grocery item is higher or lower than the first. S/he subsequently guesses if the third grocery item is higher or lower than the second. For each correct guess, s/he wins another roll, for a maximum possibility of three rolls of the dice. The five dice are exactly the same; each has a car picture on three sides of the die and dollar amounts ($500, $1000, and $1500) on the other three sides. If the contestant should roll cars on all five dice, s/he wins the car. Should the roll have at least one die not showing a car, the contestant can take the money shown on the cash die/dice and leave the game, or "freeze" the car die/dice and roll the remaining dice (should s/he have rolls remaining). If the contestant obtains five dice with cars by the end of his or her rolls, s/he wins the car; if not, s/he wins the total dollar amount shown on the dice on the last turn.

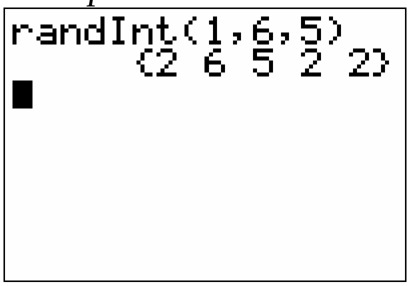
Set-up: Each student should have a TI 82, TI 83 or TI 83+ calculator

Learning objectives: probability through simulation, complement rule, binomial random variables

Let 'Em Roll
(Simulation using TI-8*)

Let 1,2,3 represent the sides with CAR on them, 4,5,6 the money.

Use the function randInt(1,6,5) to simulate one toss of the 5 dice.
Example:



For this trial, we have CAR on the first die, money on the second and third and CAR on the fourth and fifth die. We have not won the car.

Repeat this 10 times (no need to type everything in again, just hit [ENTER]).

- How many times did you win a car?

- Now pool the class results: #trials _________

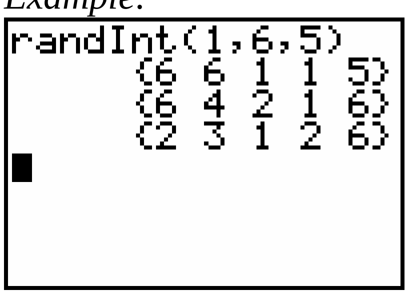
  # cars won _________

  P(winning a car with one toss of all 5 dice) _________

____

Now let's assume you have all three throws of the 5 dice.

Use the TI calculator to simulate three throws of the dice: type randInt(1,6,5), [ENTER], [ENTER], [ENTER].
Example:



For this example, I got one of the CAR dice in positions 1 & 2 (on the third throw), in positions 3 & 4 (on the first throw) but not at all in position 5. So, I didn't win the car (but I got close!).

Play the game 10 times. Each time, record the number of CARs you have after each roll. NOTE: Since you keep each CAR that you had previously, this number should never go down!

	Number of dice that say CAR…		
Game #	…after 1st roll	…after 2nd roll	…after 3rd roll
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

- What proportion of your dice *eventually* said CAR?

- Now pool the class results: # of trials (#students x 10) _________

  # of dice that eventually said CAR _________

  P(a die eventually says car) _________

- How many times did you win a car?

- Now pool the class results: # of trials (#students x 10) _________

  # cars won _________

  P(winning a car with three throws of the 5 dice) _________

Now for the theory:

- If you roll one die once, what is the probability you roll a car?

- If you roll one die 3 times, what is the probability you never roll a car?

- What is the probability that, in 3 rolls of one die, you will eventually roll a car?

- Let x represent the number of dice (out of the 5) in *Let 'Em Roll* that eventually show a car after 3 rolls. What is the probability distribution of x?

- What is the probability that all 5 dice eventually show a car? (That is, what is the chance of winning *Let 'Em Roll*?)

Who Wants to Be a Millionaire?

Game description: A contestant answers trivia questions of increasing difficulty, with four multiple choice options available on each question. Money amounts increase as the questions proceed: the first question earns the contestant $100, while the 15th and final question is worth $1 million. At any point, the contestant may read a question, choose to stop, and take home the money s/he has thus far earned. If the contestant ever chooses a wrong answer, the game ends and the contestant leaves with a cash amount dependent upon the question missed ($0 for questions 1-5, $1,000 for questions 6-10, and $32,000 for questions 11-15). To assist, the contestant has three "lifelines": elimination of 2 of the 3 wrong choices; an audience poll; a telephone call to a friend.

Set-up: None.

Learning objectives: formulating hypotheses, evidence against a null hypothesis, probability value (*P*-value)

Who Wants to Be a Millionaire?

- Without the use of any lifelines, what is the chance the contestant can <u>guess</u> the correct answer to a question?

- Without the use of any lifelines, what is the chance the contestant can guess two correct answers in a row?

- Fill in the following chart.

Number of correct guesses in a row	Probability…	
	…as a fraction	…as a decimal
1		
2		
3		
4		
5		
6		

- How many questions does a contestant have to get correct in a row before <u>you</u> believe s/he is not guessing?

Suppose we want to prove that the contestant is <u>not</u> guessing.

- Write the *null hypothesis* and the *alternative hypothesis* in words.

- You see a contestant answer 6 questions correctly in a row. <u>Assuming the null hypothesis is true</u>, what is the probability this would occur? (Hint: use your table.)

- In the previous question, you assumed the null hypothesis was true. Based on the probability you gave, do you find this assumption plausible? If not, what's the alternative?

- Let *p* stand for the percent of <u>all</u> *Millionaire* questions this contestant would get right. If the contestant is guessing, what is the value of *p*?

- Write the null and alternative hypotheses formally, in terms of *p*.

Master Key (The Price Is Right)

<u>Game description</u>: Master Key is a unique game played for three prizes; a three-digit prize, a four-digit prize, and a car. The game begins with the contestant guessing the prices of two small two-digit items. The contestant is shown a three numbers for each of these items, for example “359.” The contestant must then decide if the first two numbers are right or if the last two numbers are right (using the prior example, is the prize $35 or $59?) If the contestant guesses right, s/he earns a key; a maximum of two keys can be earned. There are five keys to choose from. Three of the keys open only one of the individual prizes (one key for the three-digit item, one key for the four-digit item, one for the car). One of the keys doesn’t open any of the prizes. The fifth and final key, the “Master Key,” opens all three prizes. The contestant tries the key(s) s/he has earned in the locks and wins whatever prizes are “opened.”

<u>Set-up</u>: None.

<u>Learning objectives</u>: combinations, probability as relative frequency, conditional probability as information, conditional v. total probability, reversal of probability (Bayes’ Rule)

Master Key

There are 5 keys available: the keys for Prize #1, Prize #2, and the car; the Master Key, which opens all 3 locks; and the "dud" key, which opens nothing. Call these keys 1, 2, Car, Master, and Dud.

Suppose a contestant wins one key.

- What is the probability this key opens the lock for the car?

The contestant tries the key in the lock for Prize #1, but the key doesn't open the lock.

- Conditional on this information, which of the original 5 keys could the contestant still have?

- What is the probability the contestant has a key that unlocks a car?

The contestant tries the key in the lock for Prize #2, but the key doesn't open that lock, either.

- Conditional on this new information, which of the original 5 keys could the contestant still have?

- What is the probability the contestant has a key that unlocks the car?

Now, suppose the contestant has won 2 keys.

- How many possible pairs of keys can the contestant choose from the original 5 keys?

- Write out all these possible pairs.

- What is the probability the contestant has at least one key that will open the lock for the car?

The contestant tries his/her first key in the lock for Prize #1, but the key doesn't open the lock.

- Conditional on this information, which of the original pairs of keys could the contestant still have?

- What is the probability the contestant has a key that unlocks a car?

The contestant tries the same key in the lock for Prize #2, but the key doesn't open that lock, either.

- Conditional on this new information, which of the original pairs of keys could the contestant still have?

- What is the probability the contestant has a key that unlocks the car?

Master Key, Part II

Now we consider the contestant’s ability to price the two small prizes. For notational purposes, let p = P(winning a key).

Consider the following questions:

a) What will be the value of p if there is no knowledge of the product price and the contestant is merely guessing?

b) What will be the value of p if the contestant has “perfect” knowledge and always prices items correctly?

c) Suppose p = 0.5. What is the probability that the contestant wins no prize? (You first have to consider how many keys s/he will win, then the chances of a key earning a prize.)

d) What is the probability that you win no prize <u>as a function of p</u>?

Now consider the following disjoint events:
A = win no prizes
B = win a prize but not the car
C = win the car

e) What is the probability that you win a prize but not the car?

f) What is the probability that you win a car?

g) Given you won a car, what is the probability that you only have one key?

Spelling Bee (*The Price Is Right*)

Game description: The contestant is shown a board with 30 cards on it; 11 of the cards say C, 11 say A, 6 say R, and 2 say CAR. The contestant picks two cards for free at the beginning of the game. The contestant can earn up to three additional picks by guessing the prices of small items. If the contestant's guess is within $10 above or below the actual price of the small item, s/he wins the item and another pick. An exact bid on any of the items wins all three items and all three picks (even if the contestant did not earn a pick on a previous item). Once the small item bidding is over, the object of the game is to spell the word "car." Each card does have a $500 cash value and the contestant always has the chance to quit the game and keep the cash shown. If the contestant spells "car," s/he wins. If the contestant finds a CAR card, s/he automatically wins.

Set-up: Have students make 30 cards (e.g., 3"x5" cards): 11 Cs, 11 As, 6 Rs, and 2 CARs. The students will simulate *Spelling Bee* by shuffling the cards and randomly selecting 5 cards. NB: They should not number the backs of the cards!

Alternatively, students can also use playing cards: 11 hearts, 11 diamonds, 6 clubs, and the 2 jokers represent Cs, As, Rs, and CARs, respectively. Again, students simulate by shuffling the cards and randomly selecting 5.

Learning objectives: probability by simulation, combinations, combinatorics, hypergeometric probabilities

Spelling Bee

Shuffle the 30 *Spelling Bee* cards. Randomly select 5 cards. If you spell "car" or select a CAR card, you win!

Play the game many times. Each time, record the number of Cs, As, Rs, and CARs you select. Also, record whether or not you win the game.

Game #	C	A	R	CAR	Win? (y/n)
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					

- What proportion of the deck of 30 cards are of Cs, As, Rs, and CARs?

card	C	A	R	CAR
Proportion of the 30 cards				

- What proportion of the cards you selected were Cs, As, Rs, and CARs?

card	C	A	R	CAR
Proportion of your cards				

- Why don't these proportions exactly match the deck?

- How many times did you win by having a CAR card? What proportion of the time did you win with this card?

  # times ___________________ proportion ___________________

- How many times did you win by having the 3 cards (C, A and R)? What proportion of the time did you win with these 3 cards?

  # times ___________________ proportion ___________________

- Pool the class results:

Total # trials ______________

Total times winning a car ______________

# of times won with CAR card ______________

# of times won with the C, A and R cards ______________

- Are you more likely to win with the one card (with CAR) or from the 3 cards (C, A and R)?

Suppose you select 5 of the 30 numbers from the *Spelling Bee* board.

- How many possible groups of 5 numbers could you have? (Hint: Does order matter?)

- Twice in the history of *The Price Is Right*, a contestant has spelled CAR three times with 5 cards: C A R CAR CAR! What is the probability of this amazing event?

- Write out all the different ways you can win *Spelling Bee*. For example, C A R CAR CAR is one way, but so is C C R A C. (Again, keep in mind: Does order matter?)

- Find the probability of each of these options.

- Add up your answers above to find the probability of winning *Spelling Bee*.

Bonus activity: Find the probability of winning *Spelling Bee* if you only select 4 of the 30 numbers.

MONTE'S DILEMMA

This game is based on the old television show "Let's Make a Deal" hosted by Monte Hall. At the end of each show, the contestant who had won the money was invited to choose from among 3 doors: Door #1, Door #2, and Door #3. Behind one of the three doors was a very nice prize, let's say a car. Behind the other 2 doors there was a goat. The contestant selected a door. Monte then revealed what was behind one of the OTHER doors (always a goat). The contestant now had a choice of 2 doors and had to choose between the one he had already chosen and the other one; in other words, he had to switch his choice. He won what was behind his final choice of door.

Intuitively: Does it make any difference to the likelihood of winning a car if the contestant switches of not?

SIMULATION: Each pair of students should have 3 cards – either with pictures of goats on 2 and a car on the third or playing cards with an Ace (car) and 2 Jacks (goats). Have one of the partners arrange the cards and act as Monte Hall and the other as the contestant. The contestant picks a door (card). Without showing the original pick, the show host shows one of the other cards (it should always be a goat). The contestant must know decide to switch or stick. The card is shown. Do this 10 times and record the results:
Modern version: visit the web site:
http://www.stat.sc.edu/~west/javahtml/LetsMakeaDeal.html

Trial	Stick/Switch?	Win/lose?
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

In how many of these trials did you win the car?

Pool the class results if necessary to answer the following:

trials switched ____________ # wins of car after switching _____________

trials stuck ____________ # wins of car after sticking _____________

Consider a tree diagram. For the first card the two options are “CAR” or “GOAT”

What is the P(Goat)? What is the P(CAR)?

Let’s assume that the probability of switching is 0.5. Fill in the remaining probabilities

What is P(CAR|switched)?

Does this agree with your original opinion?