See further:

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CHAPTER XIX

PLATO IN THE ACADEMY—FORMS AND NUMBERS

TO us Plato is first and foremost a great writer, but from his own point of view, books and the study of them are a secondary interest with the "philosopher"; what counts as supreme is a life spent in the organized prosecution of discovery (τὸ συζῆν). There can be no doubt that Plato thought his work as the organizer of the Academy much more important than the writing of dialogues. Since Aristotle commonly refers to the teaching given in the Academy as Plato's "unwritten doctrine" (ἄγραφα δόγματα), we may be reasonably sure that Plato did not even prepare a MS. of his discourses. This explains why there were several different versions in the next generation of the famous lecture on "the Good," which seems to have contained Plato's most explicit account of his own philosophy. We are told that several of the hearers, including Aristotle, Xenocrates, and Heraclides of Pontus, all published their notes of it, and the obvious implication is that there was no "author's MS." to publish. Consequently we have to discover Plato's ultimate metaphysical positions indirectly from references to them in Aristotle, supplemented by occasional brief excerpts, preserved by later Aristotelian commentators, from the statements of Academic contemporaries of Aristotle, like Xenocrates and Hermodorus. This creates a serious difficulty. When it is a mere question of what Plato *said*, the testimony of Aristotle is surely unimpeachable; but when we go on to ask what Plato *meant*, the case is different. Aristotle's references are all polemical, and Aristotle is a controversialist who is not unduly anxious to be "sympathetic." Unfortunately, too, mathematics, the science specially important for its influence on Plato's thought, is the one science where Aristotle shows himself least at home. Thus there is always the possibility that his criticisms may rest on misunderstanding. And the misunderstandings may not even originate with him. The criticism of Plato all through the *Metaphysics* seems to be subsidiary to Aristotle's standing polemic against Xenocrates, the contemporary head of the Academy. Hence it is possible that much of the criticism of *Metaphysics* M–N, the most sustained anti-Academic polemic in Aristotle, may be directed rather against Academic misinterpretation of Plato than against Plato himself.

In a necessarily brief statement our safest course is to deal only with views expressly attributed by Aristotle to Plato, and with them only so far as their meaning seems to be beyond reasonable doubt. This is, at any rate, all I can attempt in the space at my disposal. But we must carefully avoid the nineteenth-century mistake of treating the statements described by Aristotle under the name of the "doctrine" (πραγματεία) of Plato as a sort of senile dotage. Aristotle definitely identifies Platonism with these doctrines and never even hints that he knew of any other Platonism, though he does occasionally remark that the dialogues differ from the "unwritten" discourses. It seems to follow that the theories called Plato's by Aristotle must have been formulated as early as 367 B.C., the year of Aristotle's entry into the Academy, and, quite possibly, even earlier.

When we turn to these Aristotelian statements we find that, for the most part, they amount to a version of the theory of forms with a very individual character, and of a much more developed type than anything the dialogues have ascribed to Socrates. There are also one or two other notices of specific peculiarities of Plato's doctrines, all concerned with points of mathematics, and it is with some of these I propose to begin, as they may help us to understand the point of view from which the doctrine of forms as known to Aristotle was formulated.

We must remember that though mathematics was by no means the only science cultivated in the Academy, it was that which appealed most to Plato himself, and that in which the Academy exercised the most thoroughgoing influence on later developments. All the chief writers of geometrical textbooks known to us between the foundation of the Academy and the rise of the scientific schools of Alexandria belong to the Academy. In Plato's own lifetime, Theaetetus had completed the edifice of elementary solid geometry, by discovering the inscription of the octahedron and icosahedron in the sphere. He and Eudoxus and others had laid the foundations of the doctrine of quadratic surds as worked out in the tenth book of Euclid's *Elements*; Eudoxus had invented the method of approximating to the lengths and areas of curves by exhaustion (the ancient equivalent of the Integral Calculus), and had recast the whole doctrine of ratio and proportion in the form in which we now have it in Euclid's fifth book, for the purpose of making it applicable to "incommensurables." We naturally expect to find traces in Plato's doctrine of this special preoccupation with the philosophy of mathematics which is characteristic of the work of the school.[1]

To understand the motives which were prompting the Academy to a reconstruction of the philosophy of mathematics, we must go

[1] For an account of the Academic work in mathematics I may refer the reader to any of the standard works on the history of mathematics, *e.g.* Zeuthen, *Histoire des mathématiques dans l'antiquité et le moyen âge* (Fr. tr., Paris, 1902), or, for a still briefer account, Heiberg, *Mathematics in Classical Antiquity* (Eng. tr., Oxford, 1922). The ancient notices are chiefly preserved in the second prologue to Proclus' *Commentary on Euclid* i., and in the scholia to Euclid.

2 flaws in Pythag. math

back to the age of Zeno. In the Pythagorean mathematics of the fifth century there were two serious logical flaws. One was that in treating geometry as an application of arithmetic, the Pythagoreans had made the point correspond to the number 1, as is indicated in the traditional definition of the point, often mentioned by Aristotle, that it is μονὰς ἔχουσα θέσιν, "a 1 with position." The identification implies the view that a point is a minimum volume, and was ruined by Zeno's acute argumentation from the possibility of unending bisection of the straight line and the impossibility of making a line longer or a volume bigger by adding a point to it. There are just two ways of meeting the difficulty: a) one is to evade it, by severing geometry from its dependence on arithmetic, as Euclid does; the other is that actually hinted at by b) Zeno's own language and definitely adopted by modern philosophical mathematicians, of making the point correspond to 0 and regarding 0, not 1, as the first of the integers.[1] It was towards this view that Plato was feeling his way, as we shall see immediately. The other great trouble was the discovery that there are "incommensurables" or "surds," *e.g.* that the ratio of the length of the side of a square to its diagonal is not that of "integer to integer." Here, again, two ways of meeting a difficulty fatal to the old philosophy of mathematics as it stood are possible. One is again to surrender the parallelism between geometry and arithmetic by admitting the existence of surd geometrical magnitudes, but denying that there are "surd" numbers. This is the position taken by Aristotle in express words and tacitly by later mathematicians like Euclid, who always represents an "incommensurable" by a line or an area. The other is that of modern rationalistic mathematics, to revise the conception of number itself, so that it becomes possible to define "irrational" numbers of various kinds and to formulate laws for their addition and multiplication in terms of the already known arithmetic of integers. The problem has only been satisfactorily solved in the work of the last half-century, but, as we saw in dealing with the *Epinomis*, this was the line which already commended itself to Plato. Geometry and "stereometry" are, according to him, really the arithmetic of the quadratic and cubic "surds," as plane geometry has been said in our own time to be simply the "algebra of complex numbers." In this way the parallelism of geometry with arithmetic is preserved by a revised and enlarged conception of arithmetic itself.[2]

1) Point correspond to number 1 led to absurdities

2) Discovery of irrationals

a)

b)

With these considerations in mind, we can readily understand certain statements which Aristotle makes about mathematical views of Plato. There are three such statements which we may at once elucidate. (*a*) Plato stated that the "point" was a

[1] Cf. the definition of the integer-series in Frege's *Grundgesetze der Arithmetik* which is, put into words, "the integers are the successors of 0."

[2] For a real comprehension of Plato's thought it is indispensable to have a grasp of the modern logic of arithmetic. I would recommend as sufficient (but also necessary) such an exposition as that given in chap. i. (Real Variables) of Professor G. H. Hardy's *Pure Mathematics*.

Plato's rejection of geometers' def. of point.

"fiction of the geometers," and spoke, instead, of the "starting-point of the line" (*Met.* A 992*a* 20). This means, of course, that Plato rejected the conception of a point as a minimum of volume, or "unit." It has no magnitude of its own but is "the beginning" of the straight line which has such a magnitude (its length). In other words, what corresponds in arithmetic to the point is not 1 but 0, if only Greek arithmeticians had possessed a word or symbol for 0. The underlying thought is that which reappears in later Greek Platonists when they speak of a line as the "fluxion" (ῥύσις) of a point, in the very terminology Newton was later to introduce into English. We are on the track of the ideas and terminology of the inventors of what we call the Differential Calculus. It is true, of course, that this notion of an "infinitesimal" which is not quite nothing nor quite something, but a nothing in the act of turning into something, involves a logical paradox and that it has only been finally disposed of by the purification of mathematical logic, which has eliminated "infinitesimals" from the so-called Infinitesimal Calculus. But the Calculus had to be there first before its purification from bad logic could be possible, and it is hard to see how it could ever have been originated without this defective but useful conception. (b) (*Met. ibid.* 22) Plato "often used to assume his indivisible lines" (πολλάκις ἐτίθει τὰς ἀτόμους γραμμάς). Aristotle, who apparently distinguishes this point from the one he has just mentioned, does not explain its meaning. In the textually badly corrupt Peripatetic tract *de Lineis insecabilibus*, which appears to be a polemic of an Aristotelian of the first generation against Xenocrates, the "indivisible line" is regarded as a minimum length, and it is urged that there are insuperable geometrical difficulties about such a conception, as, in fact, there are. What Plato may have meant by the expression we can only conjecture. As a conjecture I offer the suggestion that his intention is precisely to deny the conception attributed to some Academic, apparently Xenocrates, by the Peripatetic tract. A line, however short, is "indivisible" in the sense that you cannot divide it into elements which are not themselves lines—in other words, it is a "continuum." The point makes a straight or curved line not by addition or summation, but by "flowing"; a straight or other line is not made of points in the way in which a wall is made of bricks laid end to end. (c) Plato said that "there is a first 2 and a first 3, and the numbers are not addible to one another" (*Met.* M 1083*a* 32, the one statement about numbers which is definitely attributed to Plato by name in the last two books of the *Metaphysics*). A similar point is made about the Academy generally in the *Ethics* (*E.N.* 1096*a* 17 ff.), where we are told that they held that there is no form (ἰδέα) of number, because "in numbers

b Cf. the observations of Stenzel, *Zahl u. Gestalt*, 89 ff. The technical expressions ῥεῖν, ῥύσις, the source of Newton's language about "fluents" and their "fluxions," come from the accounts of the doctrine in the Aristotelian commentators and were presumably coined by the Academy.

there is a before and an after," *i.e.* because numbers form a *series*. The meaning of these statements seems not to have been clear to Aristotle, but is manifest to anyone who has learned to think of number *en mathématicien*. The sense is that the series of numbers is not made by adding "units" together. *E.g.* we say that 3+1=4, but we do not mean that 3 is three "units" or that 4 is 3 *and* 1; 4 is not four 1's, or a 3 and a 1, it is *one* 4. What we really *add* together is not numbers but aggregates or collections. Thus it is true that if you have a group of *n* things and another group of *m* things, and form the two into one group, the new group contains *m*+*n* things, but it is not true that the *number m+n* contains a number *m* and a number *n*. The importance of this view is that it leads to revision of the whole conception of number. The fifth-century theory, still represented by Euclid's definition of ἀριθμός (*Elements* vii. def. 2) is that a "number" is πλῆθος μονάδων, a "collection of 1's." On the new view, the only really sound one, no number is a "collection"; the statement that 3=2+1, which is the definition of 3, does not mean that 3 is "a 2 and a 1," but that 3 is the term of the integer-series which comes "next after" 2. This explains why there is no form of number. The reason is that each "number" is itself a form, as was really implied in the *Phaedo* itself when Socrates spoke of "the number 2" and "the number 3" as instances of what he meant by a form. Hence the ordered series of integers is not a form, it is a series of forms. The point may be grasped if we remember that in our own philosophy of mathematics we do not find it possible to define "number" or even "integer"; all that we can do is to define the *series* of integers or the series, *e.g.*, of "real" numbers, and to define individual numbers. I can define "the integer series" as a series of a certain type with a certain first term, and I can define "the integer" *n*+1 by saying that it is the number of that series which is next after *n*, but I cannot really define "integer." Aristotle is never tired of arguing against Plato that there is no number except what Aristotle calls "mathematical" number, or alternatively "number made of 1's" (μοναδικὸς ἀριθμός); but the simple truth is that *no* "number" is "made of 1's," and that it is precisely what Aristotle calls "mathematical" number which has no existence except in his imagination. Plato may well have been led to this denial that numbers are "addible" by his recognition that "surds" like $\sqrt{2}$, $\sqrt[3]{2}$, must be admitted into arithmetic as numbers, since it is evident that no process of "adding 1 to 1" could ever yield such numbers as these.[1] Thus this doctrine, also, may well be connected with the fact that the "real" numbers form *a continuum*. But it is important to be clear on the point that the principle that number is not really generated by addition of 1's

Is this suggesting the synthetic-ity of math. (Kant)?

[1] This is the consideration made prominent in the treatment of the doctrine by M. Milhaud in *Les Philosophes-géomètres de la Grèce*, a work really indispensable to the student of Plato. But, as we shall see immediately, it is not the whole, nor the most important part, of Plato's doctrine.

applies equally to the numbers of the integer-series, which is not a continuum.

This brings us to the consideration of Aristotle's account of Plato's theory of forms. According to the *Metaphysics*,[1] Plato actually called the forms numbers, and maintained that each form or number has two constituents, the One which Aristotle regards as the formal constituent, and something called the "great-and-small" or "the indeterminate duality" (ἀόριστος δυάς), which Aristotle treats as a material constituent. In other words, a number is something which arises from the determination of a determinable, (the great-and-small), by the One. Since the forms are the causes of all other things, these constituents of the forms are the ultimate constituents of everything, and this is what is meant by the statement that other things "participate" in the forms.[2] Aristotle remarks on the theory that it is of the same type as the Pythagorean doctrine that "things are numbers," or are "imitations of numbers," but differs from that view by substituting the "duality" of the "great-and-small" for the "indefinite" (ἄπειρον) as one constituent of numbers, and also by maintaining that "mathematicals" (τὰ μαθηματικά) are intermediate between numbers and sensible things, whereas the Pythagoreans said that the numbers are the things.[3] He seems also to connect this theory with the special point in respect of which he holds Plato and the Pythagoreans inferior to Socrates, namely, that they "separated" (ἐχώρισαν) the "universals" or forms from "things" as Socrates had not done.[4]

Participation = Forms cause all things

It is plain from the explanations attempted by the later commentators on Aristotle that the chief source from which the doctrine alluded to in the *Metaphysics* was known in antiquity was the reports of the auditors of Plato's famous lecture on "the Good." As we do not possess these reports and cannot be sure how far the statements of Peripatetic commentators on Aristotle about them can be trusted, we need to be cautious in our interpretation. But there are certain points on which we can be reasonably certain. It is quite clear from the whole character of Aristotle's polemic against "ideal numbers," that the numbers which Plato declared

[1] *Met.* A 987*b* 18–25.

[2] The simple meaning of this is that, as we have been told by Timaeus, all the characters of "things" depend on the geometrical structure of their particles, and thus, in the end, on the structure of the "triangles" into which the faces of these particles can be resolved. And a triangle is determined again by three "numbers," those which give the lengths of its sides.

[3] *Met.* A 987*b* 25–28. Oddly enough, he does not mention the much more important point that the One is made by Plato the formal constituent in a number, whereas the Pythagoreans taught that "the unit" is the first *product* of the combination of *their* two constituent factors, πέρας and ἄπειρον, though he had correctly stated this doctrine just before, *Met.* A 986*a* 19.

[4] *Met.* M 1078*b* 30. Plato is not named in this passage, but a comparison of the criticism passed immediately below (1078*b* 34 ff.) with that made on Plato at A 990*b* 2 ff., shows that Aristotle regards the charge of making the "separation" as applicable to him.

to be forms are just the integers and nothing else, and also that the doctrine does not mean that it is denied that "man," "horse," and the like are forms, but that "the form of man" and the like are now held to be themselves in some sense "numbers." Hence Aristotle can raise the difficulty whether the "units" which make up the number which is the form of man or horse are the same as those which are found in the form of animal, or those of the form of man the same as those of the form of horse (*Met.* 1081*a* 9, 1082*a* 18, 1084*a* 13). It also looks as though Aristotle meant to ascribe to Plato, as well as to the Pythagoreans, the view that the integer-series is a succession of repetitions of the numbers up to 10, so that the Form-numbers would be, in a special sense, the first ten natural numbers. (*E.g. Met.* 1084*a* 12, though the allusion there might be rather to a theory of the Pythagoreans and Speusippus than to a personal view of Plato.) It seems clear, at any rate, that the key to the doctrine, if we could recover it, would be found in a theory of the character of the series of integers up to 10.

To some extent, at least, it seems possible to recover this key. We have to begin by understanding what is meant by speaking of one constituent of a number as the "great-and-small" and by calling this an "indeterminate duality." Even without the help of the commentators on Aristotle, the *Philebus* would enable us to give a reasonable answer to this question. We saw there that "that which admits of more and less indefinitely" was Plato's description of what we call a "continuum," though the number-series itself does not figure among the examples of continua given in the dialogue. This enables us to see at once why Plato spoke of what the Pythagoreans had called the "unlimited" (ἄπειρον) as a "great-and-small" or a "duality." It is a duality because it can be varied indefinitely in either of two directions. Probably the commentators are right in connecting this with the more specific view that you can equally reach plurality, starting from unity, by multiplication or by division, *e.g.* when you divide a given class regarded as a whole into sub-classes, you have two or more more determinate forms within the original γένος. This indicates a direct connexion between the theory of number ascribed to Plato by Aristotle and the preoccupation with the problem of the subdivision of forms in the later dialogues on which Stenzel has done well to insist, though he has allowed himself to neglect too much the specifically mathematical problem. We can also see, I think, why the other constituent of a number should be said to be "the one," and why the "unit" is no longer regarded, in Pythagorean fashion, as a "blend" of "limit" with the "unlimited," but as itself the "limit." Here, again, we have a point of contact with the theory of logical "division." As the *Philebus* had taught us, we may arrive at a "form" in either of two ways; we may start with several 1) different εἴδη as many and seek to reduce them to unity by showing that they are all special determinations of a more general "form," or again we may start with the more general "form" and discover 2) more specific "forms" within it; whichever route we follow, we presuppose as already familiar the notions of *a* form and of forms in the plural. "A" and "some" will be ultimate indefinables.[1]

In the case of numbers it is easy to see how the conception, already implied in the *Epinomis*, of a "continuum" of "real" numbers leads to the Platonic formulas. If we wish to discover a number whose product by itself is 2, it is easy to show that we can make steady approximation to such a number by constructing the endless "continued fraction":

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}}$$

By stopping off the fraction at successive stages, we get a number of values 1, $1+\frac{1}{2}$, $1+\cfrac{1}{2+\frac{1}{2}}$, etc., with the following peculiarities. The values are alternatively less and greater than $\sqrt{2}$, and each value differs from $\sqrt{2}$ less than the preceding value; by carrying the fraction far enough, we can get a fraction a/b such that a^2/b^2 differs from 2 by less than any magnitude we please to assign. This is what we mean by saying that $\sqrt{2}$ is the limiting value to which the fraction "converges" when it is continued "to infinity." Now in forming the successive approximate values, or "convergents," we are making closer and closer approximation to the precise determination of an "infinite great-and-small." It is "infinite" because however many steps you have taken, you never reach a fraction which, when multiplied by itself, gives exactly 2 as the product, though you are getting nearer to such a result at each step. It is "great-and-small," because the successive approximations are alternatively too small and too large. $\sqrt{2}$ is, so to say, gradually pegged down between a "too much" and a "too little," which are coming closer together all the time. I choose this particular example because this method of finding the value of what we call $\sqrt{2}$ was pretty certainly known to Plato.[2]

[1] We must, of course, distinguish carefully between the notion of "a" and that of "the integer 1." The latter is definable exactly as any other integer is. 1 is the number of any group x which satisfies the conditions that (*a*) there is an a which is an x; (*b*) "b is an x" implies "b is identical with a." This distinction is not yet clearly recognized in the Platonic formula.

[2] The denominators and numerators of the successive "convergents" are the series called in Greek respectively the πλευρικοί and the διαμετρικοὶ ἀριθμοί. The rule for finding any number of them is given by Theon of Smyrna (p. 43–44, Hiller). The geometrical construction by which the rule was discovered is given by Proclus (*Comm. in Rempubl.* ii. 24, 27–29, Kroll). The source of both Theon and Proclus appears to be the Peripatetic Adrastus in his commentary on the *Timaeus* (Kroll, *op. cit.* ii. 393 ff.). Plato himself alludes to the πλευρικοί and διαμετρικοὶ ἀριθμοί at *Rep.* 546*c* 5.

The same point might be similarly illustrated by the definitions given by modern mathematicians of the "real numbers." The definitions are to a certain point arbitrary, but they all turn on the notion of a "section." *E.g.* we cannot find a rational fraction the "square" of which is exactly 2. But we *can* divide all rational fractions into two classes, those of which the "squares" are less than 2 and those of which the "squares" are not less than 2. We see at once that the first of these sets has no highest term, the second no lowest, and that no fraction can belong either to both sets or to neither set; thus our "section" is unambiguous, *i.e.* every fraction falls into one and only one of the two sets thus constituted. We may then define the "square root of 2" either as this "section" itself, or, if we prefer it, as the set of "fractions whose squares are less (or, if we like, greater) than 2." Here again, the notion of a "section" of the rational fractions exhibits the Platonic characters. It involves a "duality," or "great-and-small," the two sets, one of which has all its terms less than, the other all greater than, a specified value, and the duality is "indefinite" because one of the sets has no highest term, the other no lowest. The section is a determination of the "great-and-small" of the fractions by the "one" precisely because it makes an unambiguous "cut" just where it does. Other cuts can be made at other places in the series, and each will define a different "real number."[1]

It is clear, however, that we have not yet exhausted the meaning of Plato's doctrine. From Aristotle's polemic we see that the Platonic analysis was not meant to apply simply to the case of the "irrationals" which Plato was the first to recognize as *numbers*. The theory also involves a doctrine of the structure of the integer-series itself, since it is clear that the numbers with which the forms are identified are, as Aristotle always assumes, the integers. The integers themselves, then, have the "great-and-small" and the "one" as their constituents. How is this to be understood?

[1] Cf. G. H. Hardy, *Pure Mathematics*², p. 14. The "rational fractions" are, to be sure, not a continuum, but they satisfy the only condition for a continuum known in Plato's time, that between any two a third can always be inserted. Stenzel rightly dwells on the connexion of the "duality" with "convergence," but misses the illustration from the πλευρικοί and διαμετρικοὶ ἀριθμοί (*Zahl u. Gestalt*, 59). The endlessness of the "continued fraction" makes it clear why the "great-and-small" was identified with the "non-being" of which we read in the *Sophistes* (Aristot. *Physics*, A 192*a* 6 ff.). The meaning of what is said about geometry, plane and solid, in the *Epinomis* will thus be, that the real scientific problem is to obtain a series of "approximations," within a "standard" which we can make as narrow as we please, to the various quadratic and cubic surds. In doing so, we are discovering the ratios of the "sides" or "edges" of the various regular polygons and solids to one another. We discover, *e.g.*, exactly how long—within a known "standard"—a line must be if the area of the square or volume of the cube on it is to be 2, 3, 5 . . . times a given area and volume; and since all rectilinear areas and volumes can be expressed as those of squares and cubes, this solves the question of the surveyor and the "stereometer." It is precisely with such metrical problems, relating to the "regular solids," that Euclid's Book XIII is concerned, a safe indication of its Academic *provenance*.

The difficulty is that the integers do not form a continuum, even in the sense in which continuity means no more than infinite divisibility, *i.e.* the possibility of inserting a third term between any two given terms of the series. For each integer is "next after" another.

How, then, does Plato suppose the series of integers to be constructed? I doubt if the notices preserved to us enable us to answer the question finally. What is clear is that Plato rightly rejects the view retained by Aristotle, that an integer is a collection of "1's," and that the series is thus constructed by additions of 1 to itself. 2 is not "1 and 1" but "the number next after 1." (This ought to be plain from the simple consideration of the way in which we learn to count. We do not count, "one, one, one, one, . . ." but "one, two, three . . .") But when we ask in what way the "duality" comes in in constructing the series of integers, we are puzzled by the confusion which seems to run through Aristotle and his commentators between the "indeterminate duality" or "great-and-small" and the *number* 2. If it were only in the polemic of Aristotle that this confusion were found, we might conceivably dismiss it as a mere misunderstanding, but it appears to have occurred also in the Academic reports of Plato's doctrine. The complete study of the problem would require a long discussion of the mass of material collected and examined by M. Robin in his volume *La Théorie platonicienne*. Here it must be enough to remark that the following points seem to be quite certain. (1) The "dyad" was called δυοποιός, because it "doubles" everything it "lays hold of." There is no doubt that the "dyad" meant is the "great-and-small," but "it also seems clear that there is a confusion, perhaps from the very first, with the αὐτὸ ὅ ἐστι δυάς, the number 2, and that the function of the "dyad" within the integer-series is thought of as being to produce the series of "powers" of 2 by repeated multiplication, 1×2, $1 \times 2 \times 2$, $1 \times 2 \times 2 \times 2$, and so forth (cf. *Epinomis* 991*a* 1–4).[1] (2) The "one," we are told, puts a stop to the "indeterminateness" of the "great-and-small" by "equalizing" or "stabilizing" it (τῷ ἰσάζειν).[2] This, I suggest, as my conjectural explanation of an obscure expression, means that each odd number is the arithmetical mean between the preceding and following even numbers, and so "halves their difference." Each odd number will be got by halving "the sum of two even" numbers. Thus the order of the "decade" will be, 1, 2, 4, 8; 3 (which equalizes 2 and 4); 6 (double of 3); 5, 7 (which "equalize" 4 and 6, and 6 and 8); 10 (double of 5); 9 (which equalizes 8 and 10).[3] Cp. Aristotle's

[1] Cf. Aristot. *Met.* 1084*a* 5, 1091*a* 12, 1082*a* 14, 987*b* 33.

[2] Plutarch, *de Anim. procreat.* 1012*d*, reporting the explanation of Xenocrates, ἐκ δὲ τούτων γένέσθαι τὸν ἀριθμὸν τοῦ ἑνὸς ὁρίζοντος τὸ πλῆθος, Aristot. *Met. M.* 1083*b* 23, 29, where the "unit" is said to arise from the "equalizing" of the "dyad" of the great-and-small.

[3] See Robin, *La Théorie platonicienne*, p. 449. The mathematical reader will see at once a certain analogy between this procedure and the "quadrilateral construction" of von Staudt.

use of the "arithmetical mean" as an "equalizer," *E.N.* 1132*a* 1 ff. If this was the construction, it must be pronounced very faulty. Not only does it involve the confusions of "a" with 1 and of "plurality" with 2, but it involves obtaining the terms of the series in an unnatural order and using more than one principle of construction where one is sufficient. (The one really satisfactory way of defining the integers is to proceed by "mathematical induction," *i.e.* to define each in terms of its immediate precursor. This is readily done in the following way. When we have defined the integer n, we can go on to define $n+1$ by the statement that $n+1$ is the number of members of a group satisfying the conditions (*a*) that it contains a group with n members, (*b*) that it contains a member a which is not a member of this group; (*c*) that it does not contain any member which is neither a nor a member of the group of n members already mentioned.)

If, as seems probable, Plato's conception has these defects, we must not be surprised. He probably started with the right conviction that what we should call the notion of a "section" is necessary for the definition of the "irrationals," and went on to extend the conception to cover the case of the integers. What could not be expected of the first thinker who had formed the notion of a "real" number is the recognition that integers, rational fractions, real numbers, do not form a single series, in other words that the "integer," 2, the "rational number" 2/1, and the "real number 2" are all distinct. In the logical construction of the types of number, we need three distinct steps: the rules for defining the successive integers, the derivation of the rational numbers from the integers, and the derivation of the "continuum" of the real numbers from the series of rational numbers. These, however, are matters on which mathematical philosophers have only reached clear comprehension in very recent times. The important point is that Plato should have grasped the necessity of enlarging the traditional conception of number and of strictly defining numbers of all kinds.[1]

What are the "mathematicals" which Plato distinguished from his numbers or forms? Aristotle tells us that they differ from forms in the fact that they are many, whereas the form is one, and from sensible things by being eternal (*Met.* A 987*b* 15). It is to be noted that he does not call them "mathematical numbers,"

[1] Stenzel, *Zahl u. Gestalt*, 31, gives a different construction, but without justifying it. I venture to think he has been misled by an anxiety to discover Plato's number theory directly in the *Philebus*, where it could not have been introduced without the dramatic absurdity of putting it into the mouth of Socrates. In the main, I hope I am in accord with Burnet, *Greek Philosophy, Part I.*, 320 ff. But I should say that I can make nothing of n. 2 to p. 320, which manifestly is a *non-sens*. It appears to be a *partially* correct explanation of something Aristotle tells us about the Pythagoreans, which has got into its present place by some inadvertence. How *can* "the one" be the terms of the series $\sqrt{2}$, $\sqrt{6}$, $\sqrt{12}$ . . . ?

but τὰ μαθηματικά, and that he never appears to ascribe to *Plato* the recognition of "mathematical *number*." The meaning seems to me to be best shown by two passages in the Aristotelian *corpus*. At *Metaphysics* K 1059*b* 2 ff., it is made an objection to the theory of forms that just as the μαθηματικά are intermediate between the form and sensible things, so there ought to be—on the theory—something intermediate between such a form as man or horse and visible men and horses (though we see that there is not). This implies that the "mathematicals" are something quite familiar. I would couple with this *de Anima* A 404*b* 19, where we are told that in τὰ περὶ φιλοσοφίας λεγόμενα Plato said that the form of animal is composed of the one and "the first length, breadth, and depth." The form of animal is, according to the *Timaeus*, the archetype on which the sensible world is constructed, that is, it is the *res extensa*, the subject-matter of geometry, and Aristotle's meaning is thus that this *res extensa* is constituted by the three dimensions of length, breadth, and depth. These *correspond*, as the context of the passage in the *de Anima* makes clear, to the numbers 2, 3, 4 (the line being determined by two points, the plane by three, three-dimensional space by four). Thus Plato's construction recalls the Pythagorean tetractys of the numbers 1, 2, 3, 4. But he spoke not of numbers, but of the first "length, breadth, depth." This seems to mean that though, as the *Epinomis* says, plane and solid geometry may be identified with the study of certain kinds of number, lengths, areas, volumes are not identical with numbers. The study of number provides the key to all these relations, and yet they are not themselves numbers, and the significance of number is not exhausted by its geometrical applications.

So we, too, are familiar with analytical geometry in which we study the properties of curves and surfaces by means of numerical equations. All the properties of the curves and surfaces can be discovered from these equations, but the application of equations is not confined to geometry or geometrical physics; the same methods, for example, play a prominent part in the study of economics, as when we plot out curves to show the effects of modifications of duties on the "volume" of foreign trade. In a word, I take it, the "mathematicals" are what the geometer studies.

We may now perhaps be in a position to see what is meant by the statement that the constituents of the forms are the constituents of everything. The things of the sensible world, as we have learned from the *Philebus*, are one and all in "becoming"; they are events or processes tending to the realization of a definite law and this law, Plato thinks, can be expressed in numerical form. Because these things are always "in the making," they do not exhibit permanent and absolute conformity to law of structure; if once they were "made" and finished, they would be the perfect embodiment of law of structure. And because the stuff of things is extension itself, the law thus realized would be geometrical and therefore, as we should say, be expressible in the form of an equation

Participation = expressible in the form of an equation

or equations. This is what Plato means at bottom in his own philosophy by the "participation" of the sensible in forms and by the doctrine that the στοιχεῖα of number are the στοιχεῖα of everything. (I abstain from commenting on the further numerous passages in Aristotle where the question of the relation of the ἀρχαί of geometry to those of arithmetic is raised, since these seem to form part of the polemic against Speusippus and Xenocrates, and it is not clear to me how far any of the views canvassed are meant to be directly ascribed to Plato.)

Aristotle seems, as I said, to connect his complaint about the Academic "separation" (χωρισμός) between forms and sensible things specially with the doctrine we have just been discussing. He is commonly taken to mean no more than that the Platonic form is a sort of "double" of the sensible thing, supposed to be in some "intelligible world," wholly sundered from the real world of actual life. It is hard to suppose that he could put such an interpretation on a theory which according to himself makes the στοιχεῖα of number the στοιχεῖα of everything. Hence I think Stenzel[1] is on the right track in looking for a more definite meaning in the Aristotelian criticism, and that he has rightly indicated the direction in which we should look. As he points out, one of Aristotle's chief difficulties about the "numbers" is that he holds that if "animal" is one number and "man" is another, we have to face the question whether the "units" in "animal" are part of the "units" which constitute "man" or not; (*e.g.* if you said "animal" is 2, "man" is 4, since $2 \times 2 = 4$, "man" would seem to be the same thing as "animal" taken twice over). The complaint, as Stenzel says, is not that an εἶδος is treated as something distinct from a sensible individual, but that the more universal εἴδη, the γένη as Aristotle calls them, are thought of as though they had a being distinct from that of the ἄτομον εἶδος or *infima species*. Aristotle's point is that "animal," for example, has no being except as "horse," "man," "dog," or one of the other species which can no longer be divided into sub-species. This would be, in effect, a criticism on the method of division as practised in the *Sophistes*, where it is made a rule that in summing up the result of the division into a definition, all the intermediate *differentiae* which have been employed must be recapitulated. This is a procedure condemned by Aristotle's own doctrine that a definition need only state genus and specific difference; the specific difference includes in itself all the intermediate differences. Hence, according to Stenzel, the χωρισμός of which Aristotle complains is that the Platonic account of "division" as

[1] See Stenzel, *Zahl u. Gestalt*, 133 ff., with the Aristotelian texts discussed there. The all-important passage is *Met.* Z 1037*b* 8–1038*a* 35. Aristotle urges that if, *e.g.*, you first divide animals into footed animals and animals without feet, and then divide the former into bipeds and others, the Platonic rule would require you to say that man is a "two-footed footed animal." But the determination "footed" only exists actually as contained in the more specific determinations "two-footed," "four-footed." The same problem recurs in *Met.* H 6, 1045*a* 7 ff.

the instrument of definition is fatal to the unity of the *definiendum*,[1] and, since the process is a direct outcome of the doctrine of *μέθεξις*, the defect is one which requires the doctrine of *μέθεξις* itself to be revised. (Thus Aristotle's rejection of the Platonic doctrine of forms would at bottom be based on rejection of the logical tenet that the relation of species to genus is identical with that of individual to species.) Whether this interesting interpretation is sound is, however, a question for the student of Aristotelianism rather than for an expositor of Plato.[2]

See further:

BURNET.—*Greek Philosophy, Part I.*, 312–324; *Platonism*, c. 5, 2, 7.

NATORP, P.—*Platons Ideenlehre*, 366–436.

BAEUMKER, C.—*Das Problem der Materie in der griechischen Philosophie*, 196–209.

STENZEL, J.—*Zahl und Gestalt bei Platon und Aristoteles.* (1924.)

ROBIN, L.—*La Théorie platonicienne des idées et des nombres après Aristote.* (Paris, 1908.)

MILHAUD, G.—*Les Philosophes-géomètres de la Grèce, Platon et ses prédécesseurs.* (Paris, 1900.)

TAYLOR, A. E.—*Philosophical Studies*, pp. 91–150.

THOMPSON, D'ARCY W.—"Excess and Defect" in MIND, N.S., 149.

[1] *Zahl u. Gestalt*, 126 ff.

[2] It seems clear that a definitive interpretation of Plato's main thought must start with a thorough study of the material collected in M. Robin's great work *La Théorie platonicienne.* It is time that we should make an end of the pretence of understanding Plato by ignoring the evidence or by arbitrarily reading into him the views of our own favourite modern metaphysicians. In this brief chapter I have only been able to hint at the interpretation the material suggests to myself. These hints I have tried to develop briefly in a notice of Stenzel's book in *Gnomon*, ii. 7 (July 1926), and more fully in an essay in MIND, "Forms and Numbers," with reference to the Aristotelian evidence. (See the reference given above.)

ADDENDA

P. 21, l. 18 ff. It seems necessary, in view of some criticisms, to say expressly that I regard the date 387 B.C. as a mere convenient "approximation," not as the known precise date of the founding of the Academy. And, of course, my language about the long interruption in Plato's literary activity must be understood with the qualifications (1) that I expressly decline to commit myself to an opinion about the relative order of composition of *Republic*, *Phaedo*, *Symposium*, and (2) that I never meant to exclude the possibility of a minor "occasional" violation of silence. On my own view the *Menexenus* would have to be dated c. 380–379. Understood in this "common-sense" way, the view that "roughly speaking" the dialogues earlier than the *Parmenides* and *Theaetetus* were written before the foundation of the Academy still seems to me as probable as it did to Burnet.

P. 207, l. 26 ff. The reality of Plato's own personal faith in immortality is surely put beyond doubt by the words of Ep. vii. 335*a*, "one must put genuine faith in the ancient sacred sayings which indicate that our soul is immortal, has to face a judge, and pays the gravest penalties when one has left the body," etc. (*πείθεσθ ια δὲ ὄντως ἀεὶ χρὴ τοῖς παλαιοῖς τε καὶ ἱεροῖς λόγοις, οἳ δὴ μηνύουσιν ἡμῖν ἀθάνατον ψυχὴν εἶναι δικαστάς τε ἴσχειν καὶ τίνειν τὰς μεγίστας τιμωρίας ὅταν τις ἀπαλλαχθῇ τού σώματος*).

P. 263, par. 2. It should be noted that the Glaucon of the *Symposium* is not Plato's brother, who figures in the *Republic*, since (*Symp.* 173*a*) he, like Plato himself, was a mere *παῖς* at the date of Agathon's party.

P. 263, par. 2. Professor Burnet, in the posthumous volume of lectures on *Platonism* delivered at the University of California, expresses the opinion that the *Republic* and consequently the *Timaeus* are to be given a dramatic date anterior to the Archidamian War (*Platonism*, pp. 25–26). This would, so far as I can see, be *possible* but for one consideration. It would compel us to hold that Perictione, since she was the mother of two sons who are young men before 431, was *at the very least* over a hundred years old in 366, when *Ep.* xiii refers to her as still living. This is just possible, but hardly likely, and since I am as convinced as Burnet himself of the genuineness of *Ep.* xiii., I would rather not follow him on this point.

P. 278, *n.* 1. Xenophon also (*Symp.* ii. 9) ascribes to Socrates the thesis that "woman's *nature* is not inferior to man's" (*ἡ γυναικεία φύσις οὐδὲν χείρων τῆς τοῦ ἀνδρὸς οὖσα τυγχάνει*), though she is not his equal in physical strength and intelligence (*γνώμης τε καὶ ἰσχύος δεῖται*). But he may be dependent on Plato or Aeschines, or on both.

P. 309, *n.* 1. Aeschines also in his *Alcibiades* ascribed the "erotic" temperament to Socrates, with special reference to his affection for Alcibiades. (*ἐγὼ δὲ διὰ τὸν ἔρωτα ὃν ἐτύγχανον ἐρῶν 'Αλκιβιάδου οὐδὲν*